The Coupled Three-Body Problem and Ballistic Lunar Capture

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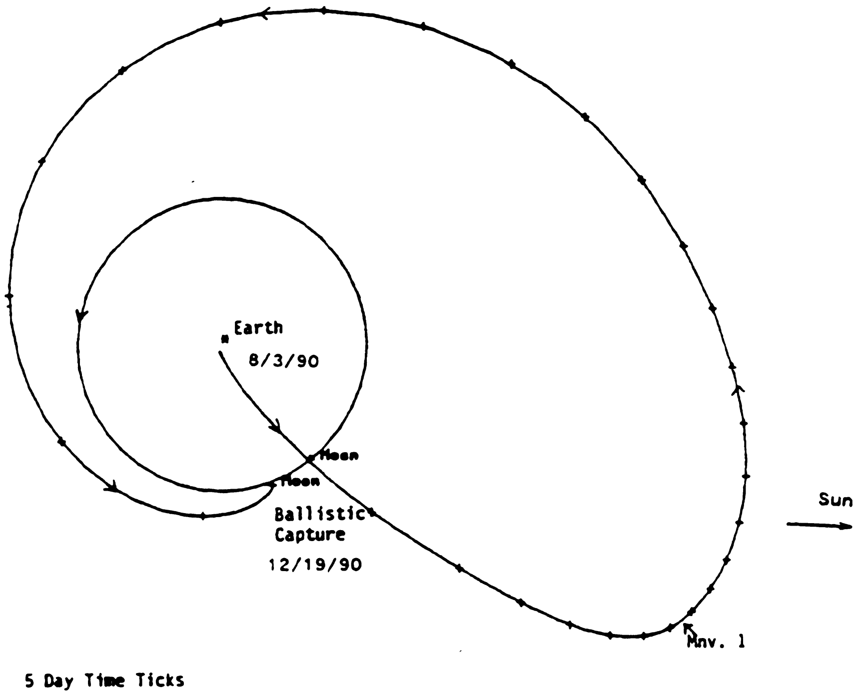
Efficient Lunar Transfer: How to "Shoot the Moon"

- Our goal: Design an efficient trajectory to the Moon using less fuel than the traditional Hohmann transfer (e.g., the Apollo missions).
- Find and classify sets of solutions to the Sun-Earth-Moon-spacecraft 4-body problem; in particular, find solutions starting from Earth and ending in ballistic capture by the Moon.
- Approximate 4-body system as two coupled 3-body systems.
- Review results on L_1 and L_2 dynamical channels in the planar circular restricted 3-body problem.
- Find intersections between dynamical channels between the two systems to construct complete Earth-to-Moon trajectory, which utilizes perturbation by Sun.

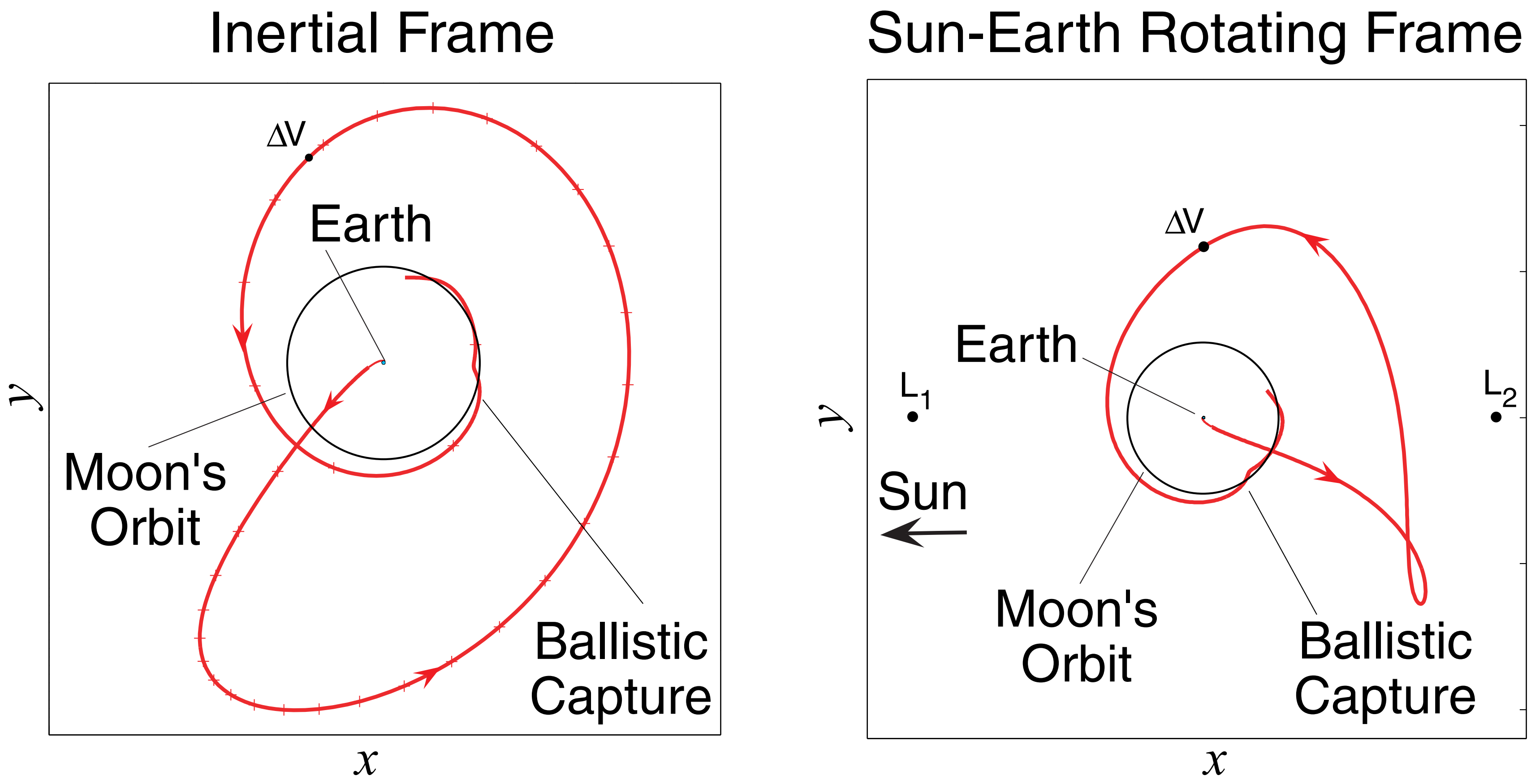
Lunar Capture: How to get to the Moon Cheaply

- In 1991, the failed Japanese mission, Muses-A (Uesugi [1986]), was given new life with a radical new mission concept and renamed as the ***Hiten Mission*** (Tanabe et al. [1982], Belbruno [1987], Belbruno and Miller [1993]).

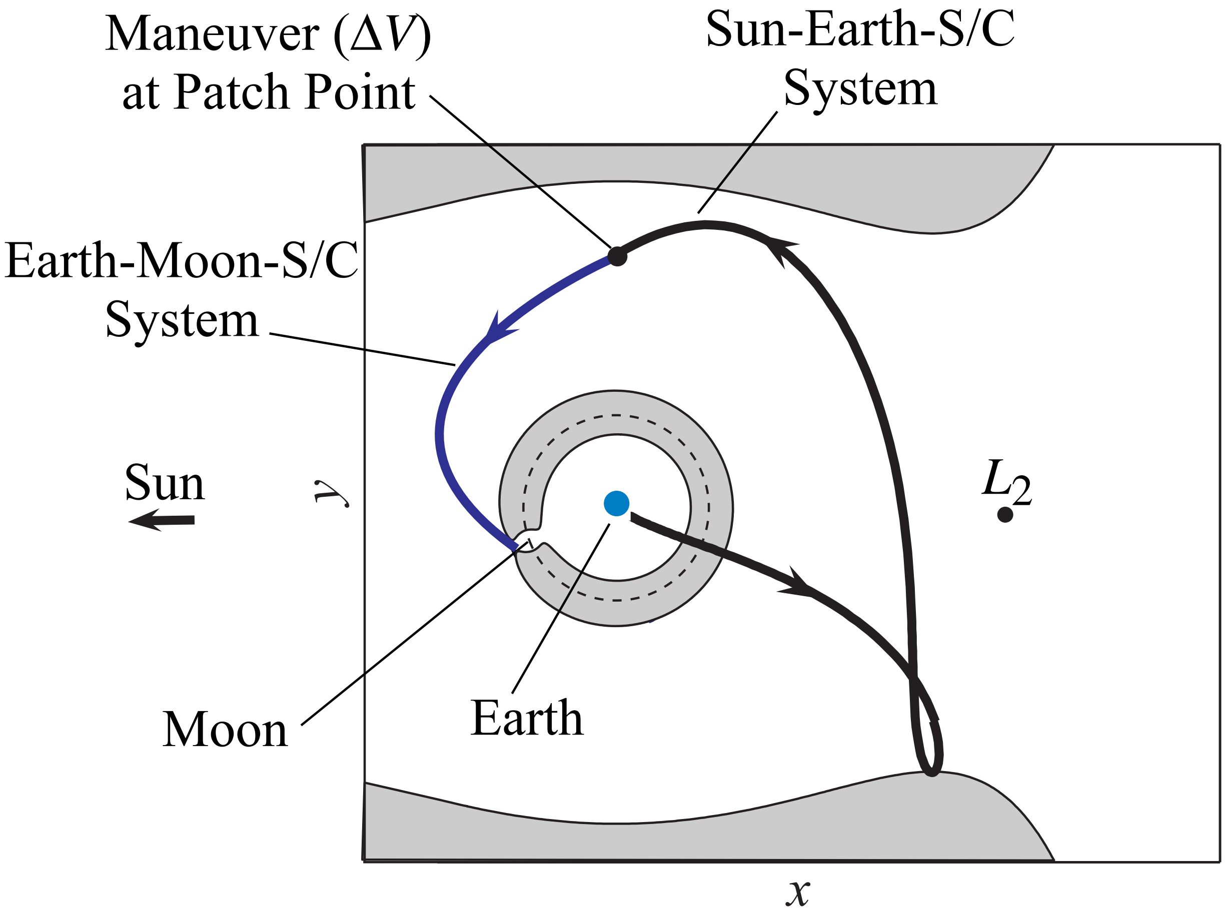


Numerical simulation of a ballistic capture transfer trajectory for the Japanese spacecraft Hiten: ecliptic plane projection, sun's direction indicated at Earth injection. (from Belbruno and Miller [1993])

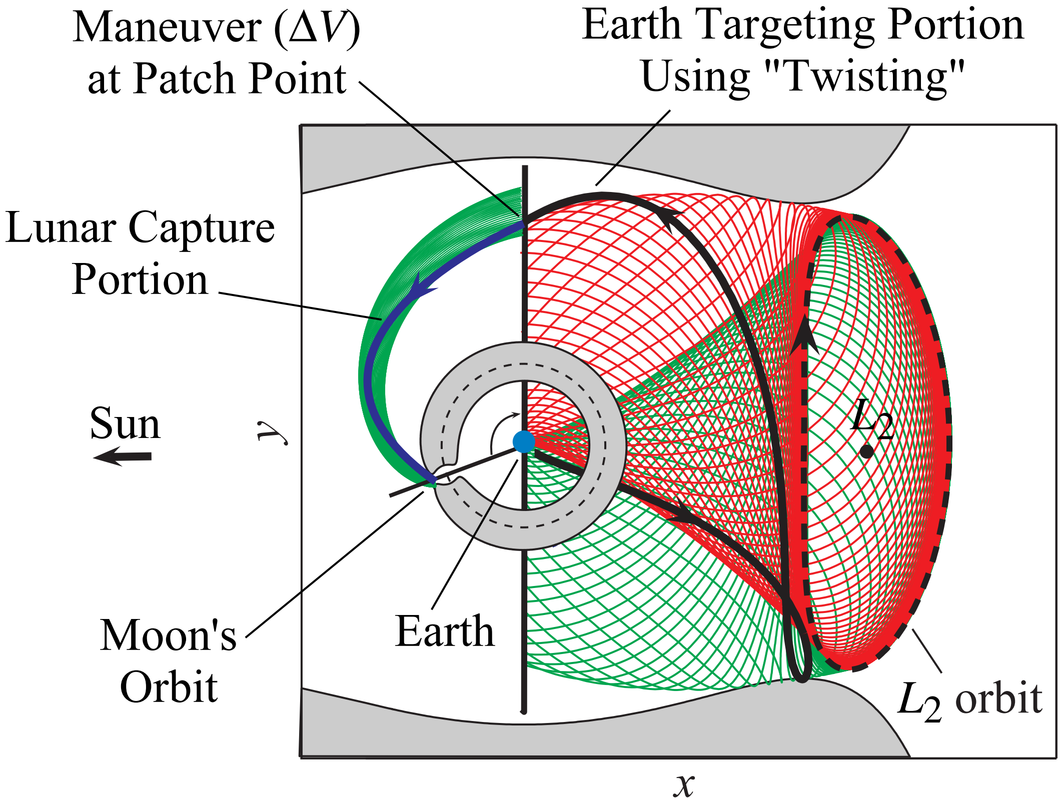
- We present an approach to the problem of the orbital dynamics of this interesting trajectory by ***implementing in a systematic way*** the view that the Sun-Earth-Moon-spacecraft 4-body system can be modeled as ***two coupled 3-body systems***.



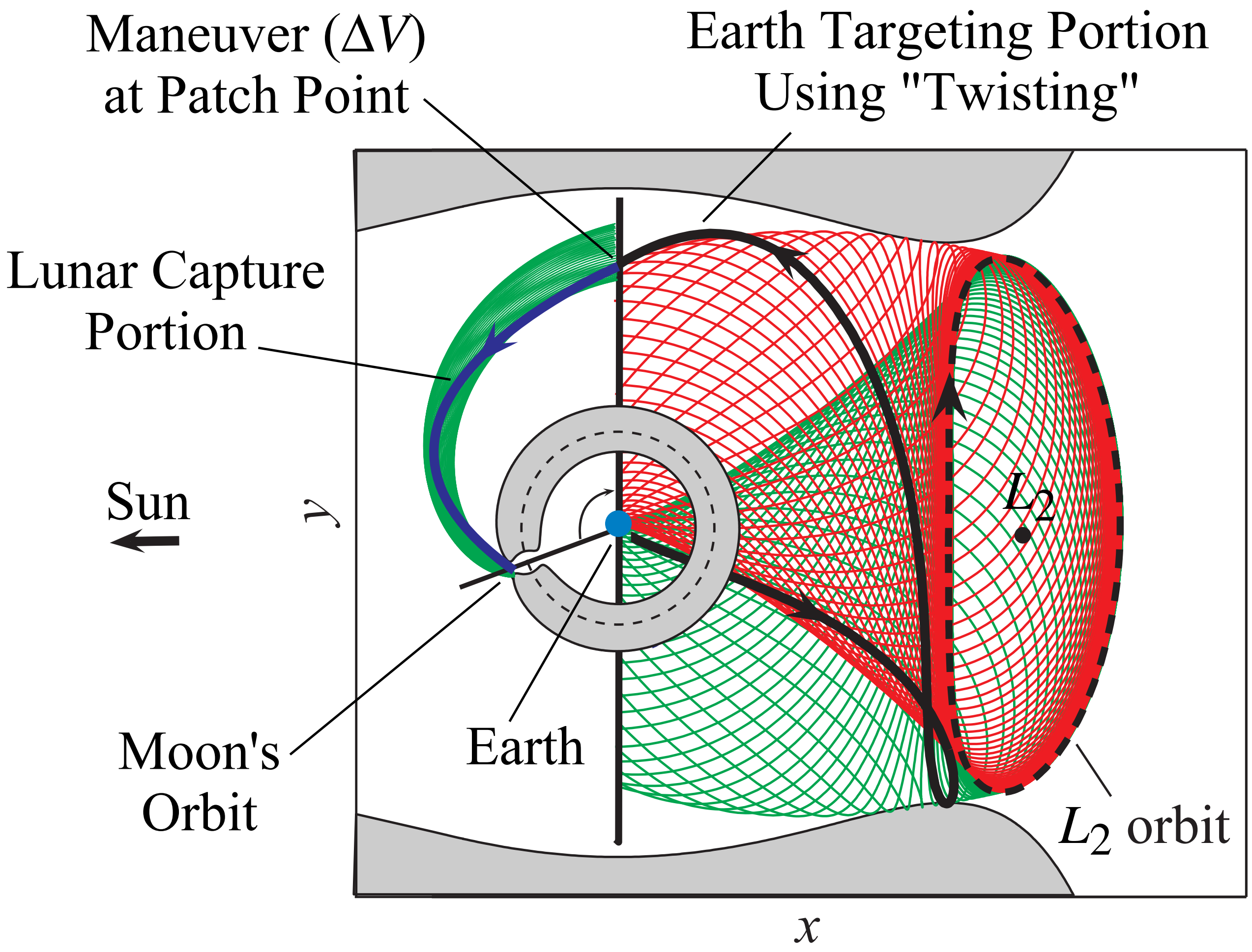
- Below is a schematic of the "Shoot the Moon" trajectory, showing the two legs of the trajectory in the Sun-Earth rotating frame:
  - **Earth backward targeting portion**
  - **Lunar capture portion**



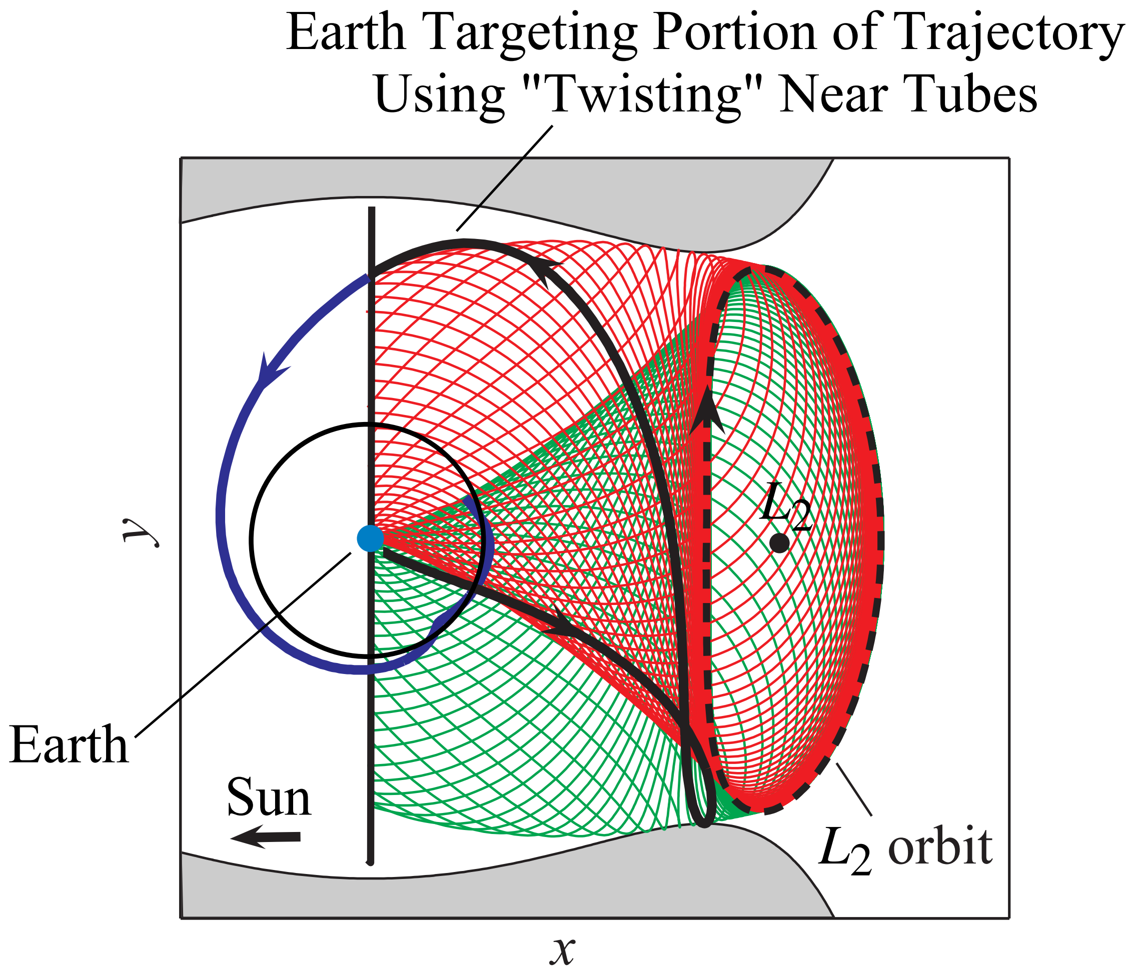
- Within each 3-body system, using our understanding of the invariant manifold structures associated with the Lagrange points L_1 and L_2, we transfer from a 200 km altitude Earth orbit into the region where the invariant manifold structure of the Sun-Earth Lagrange points interact with the invariant manifold structure of the Earth-Moon Lagrange points.



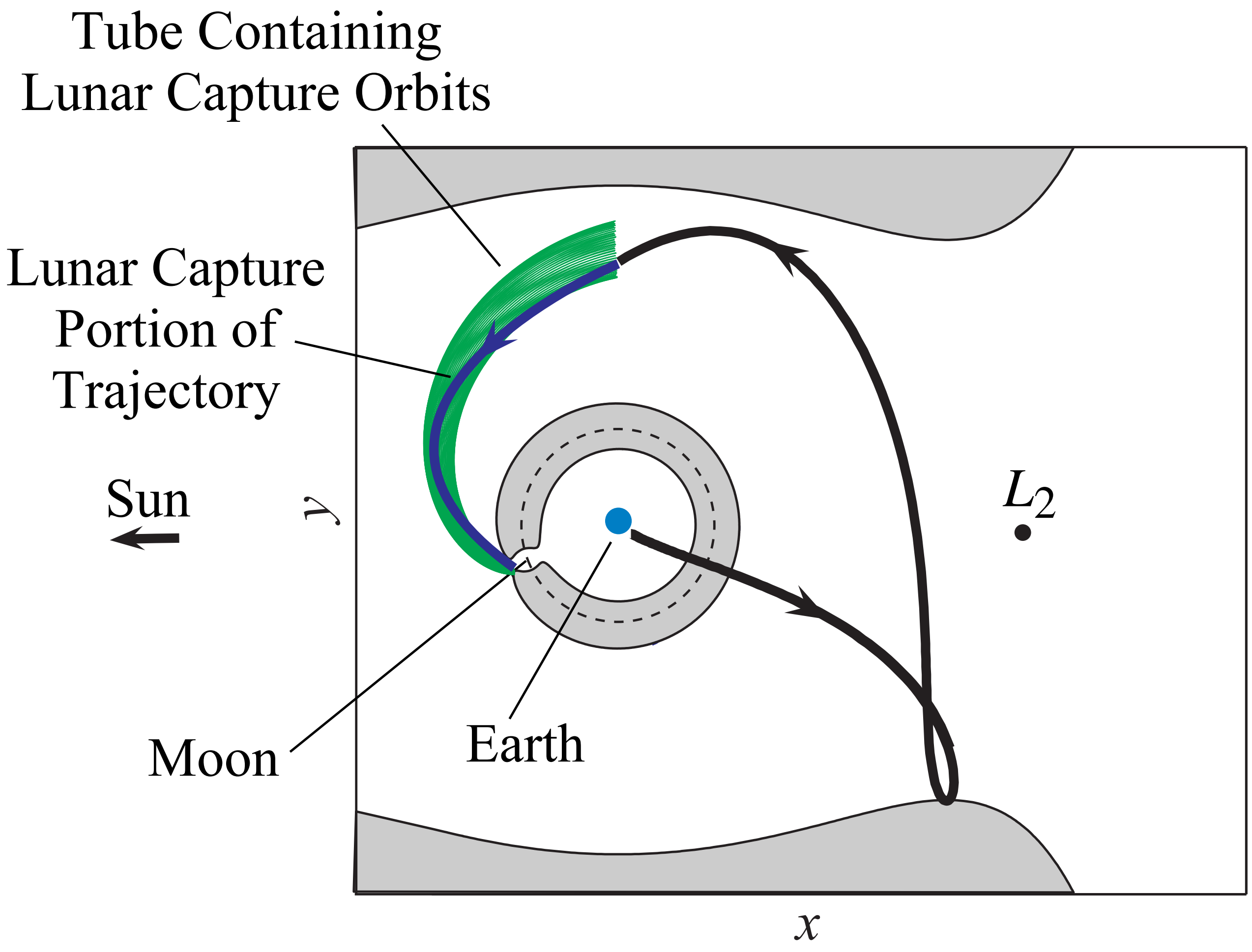
- Schematic with Lagrange point invariant manifold structures:
  - **Earth backward targeting portion**
  - **Lunar capture portion**



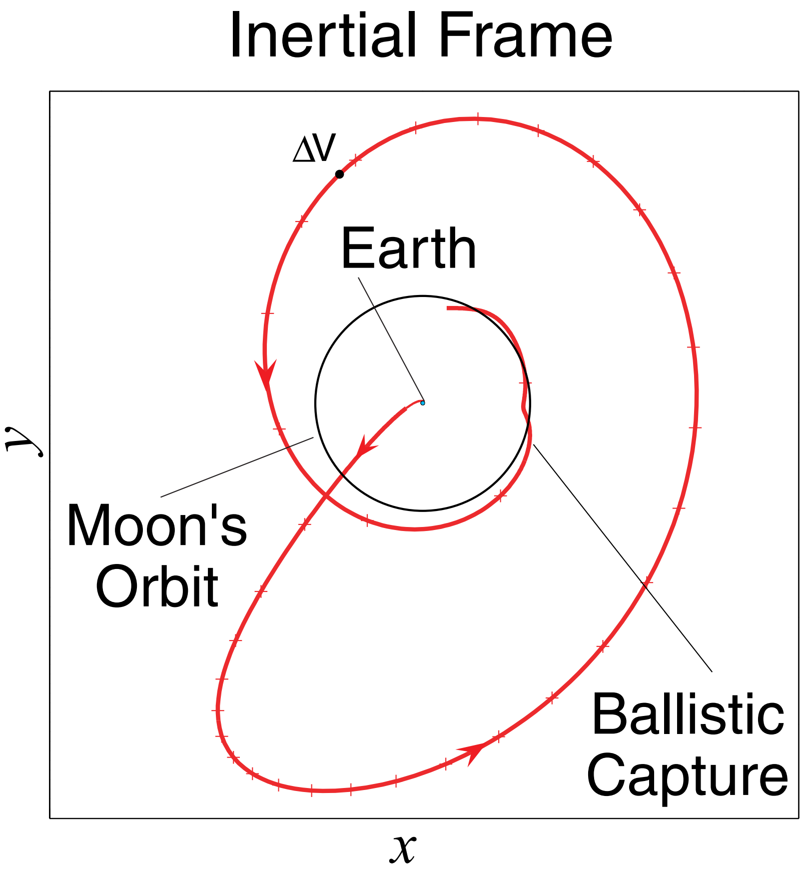
- One utilizes the sensitivity of the “twisting” near the invariant manifold tubes to target back to a suitable Earth parking orbit.

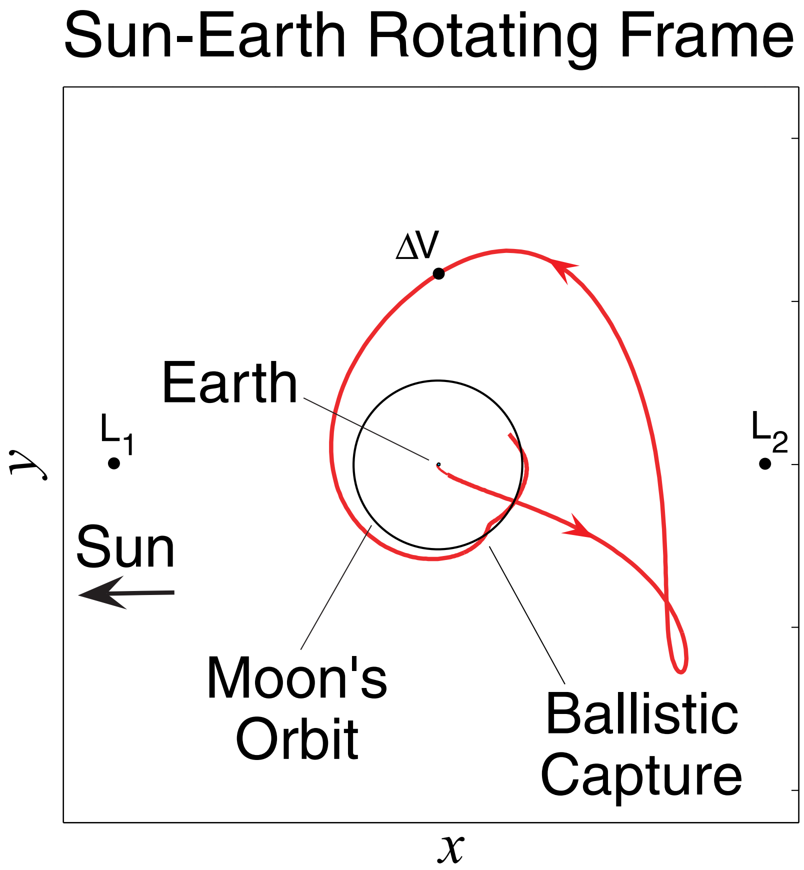


- This interaction permits a fuel efficient transfer from the Sun-Earth system to the Earth-Moon system. The invariant manifold tubes of the Earth-Moon system provide the dynamical channels in phase space that enable ballistic captures of the spacecraft by the Moon.

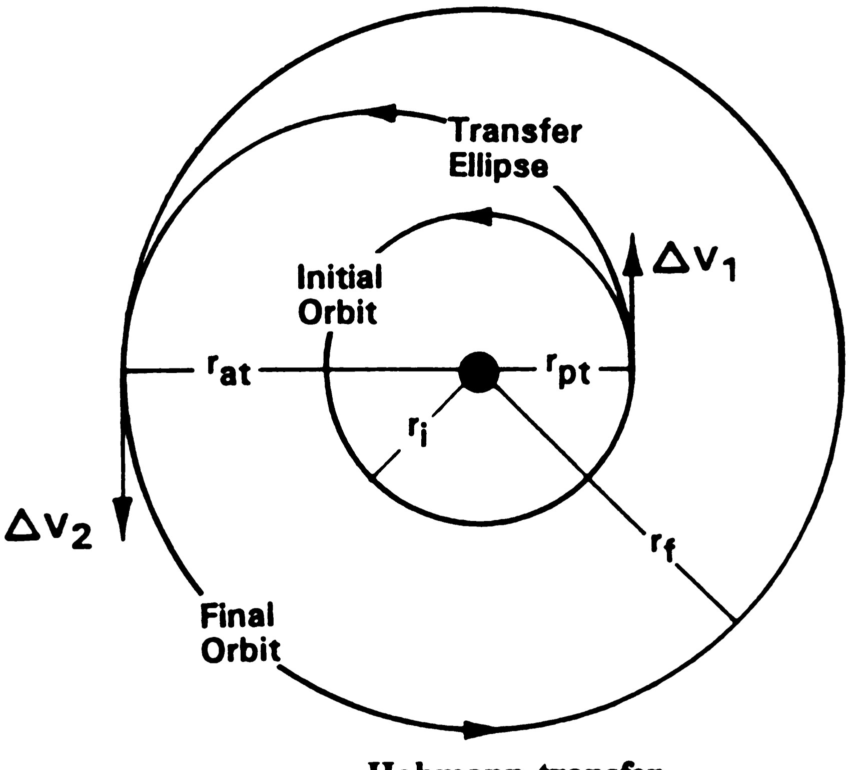


- The results are then checked by integration in the bicircular 4-body problem. It works!

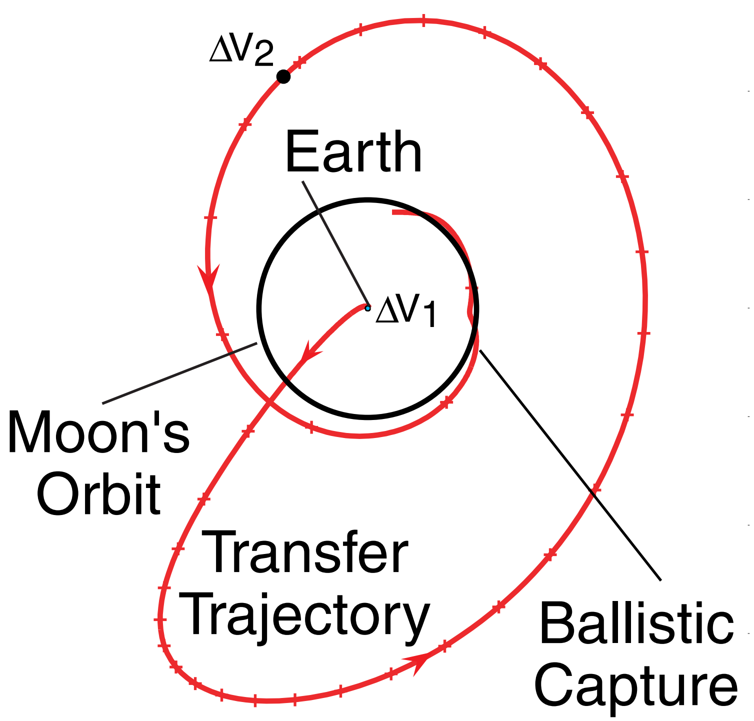




- This technique is cheaper (about 20% less ΔV) than the usual Hohmann transfer (jumping onto an ellipse that reaches the Moon, then accelerating to catch it, then circularizing). However, it also takes longer (4 to 6 months compared to 3 days).



Hohmann transfer.
(from Brown [1992])



Planar Circular Restricted Three Body Problem–PCR3BP

General Comments

- Describes the motion of a body moving in the gravitational field of two main bodies (the ***primaries***) that are moving in circles.
- The two main bodies could be the ***Sun and Earth***, or the ***Earth and Moon***, etc. The total mass is normalized to 1; they are denoted $m_S = 1 - \mu$ and $m_E = \mu$, so $0 < \mu < \frac{1}{2}$.
- Let μ be the ratio between the mass of the Earth and the mass of the Sun-Earth system,

$$\mu = \frac{m_E}{m_E + m_S},$$

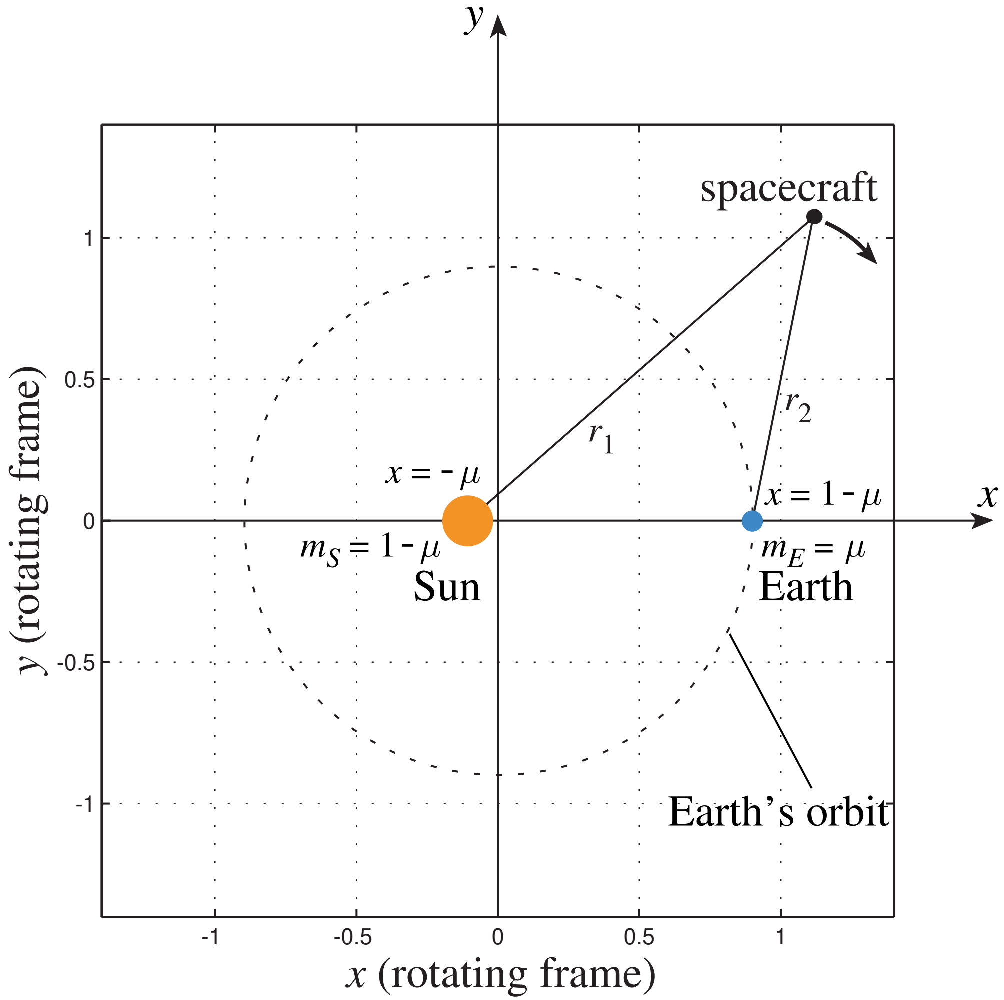
  - For the Sun-Earth system, $\mu = 3.03591 \times 10^{-6}$.
  - For the Earth-Moon system, $\mu = 0.01215$.

- The two main bodies rotate in the plane in circles counterclockwise about their common center of mass and with angular velocity ω (also normalized to 1).
- The third body, the ***spacecraft***, has mass zero and is free to move in the plane.
- The ***planar*** restricted three body problem is used for simplicity. Generalization to the ***three dimensional problem*** is of course important, but many of the effects can be described well with the planar model.

Equations of Motion

- ***Notation:*** Choose a ***rotating coordinate system*** so that
  - the origin is at the center of mass
  - the Sun and Earth are on the x-axis at the points $(-\mu, 0)$ and $(1-\mu, 0)$ respectively–i.e., the distance from the Sun to Earth is normalized to be 1.
  - Let (x, y) be the position of the comet in the plane relative to the positions of the Sun and Earth
  - distances to the Sun and Earth:

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \quad \text{and} \quad r_2 = \sqrt{(x-1+\mu)^2 + y^2}.$$



y
x
spacecraft
r_1
r_2
$x = -\mu$
$x = 1 - \mu$
$m_S = 1 - \mu$
Sun
$m_E = \mu$
Earth
Earth's orbit
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
y (rotating frame)
x (rotating frame)

- ***Equations of motion:***

$$\boxed{\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y}}$$

  where

$$\Omega = \frac{(x^2 + y^2)}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2}.$$

- They have a first integral, the ***Jacobi constant***, given by

$$C(x, y, \dot{x}, \dot{y}) = -(\dot{x}^2 + \dot{y}^2) + 2\Omega(x, y),$$

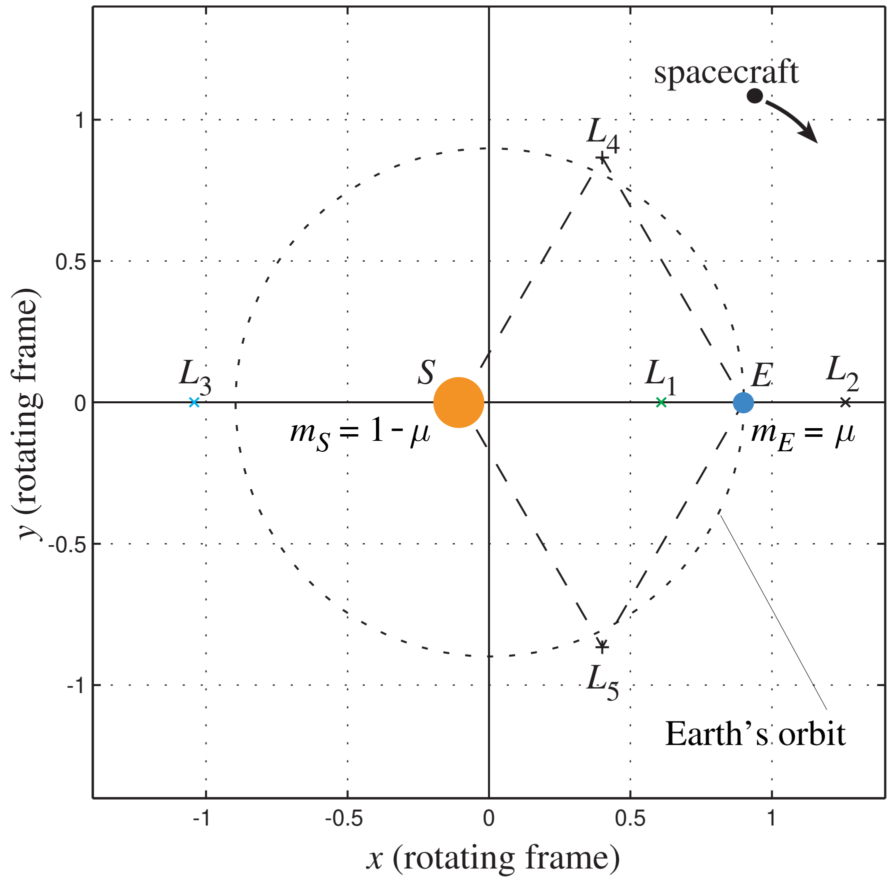
  which is related to the Hamiltonian ***energy*** by $C = -2H$.

- ***Energy manifolds*** are 3-dimensional surfaces foliating the 4-dimensional phase space.

- For fixed energy, ***Poincaré sections*** are then 2-dimensional, making visualization of intersections between sets in the phase space particularly simple.

Five Equilibrium Points

- Three ***collinear*** (Euler, 1750) on the x-axis— L_1, L_2, L_3
- Two ***equilateral points*** (Lagrange, 1760)— L_4, L_5.



Stability of Equilibria

- Eigenvalues of the linearized equations at L_1 and L_2 have one real and one imaginary pair. The ***stable and unstable manifolds*** of these equilibria play an important role.
- Associated periodic orbits are called the ***Lyapunov orbits***. Their stable and unstable manifolds are also important.
- For space mission design, the most interesting equilibria are the ***unstable*** ones, not the stable ones!

Consider the dynamics plus the control!

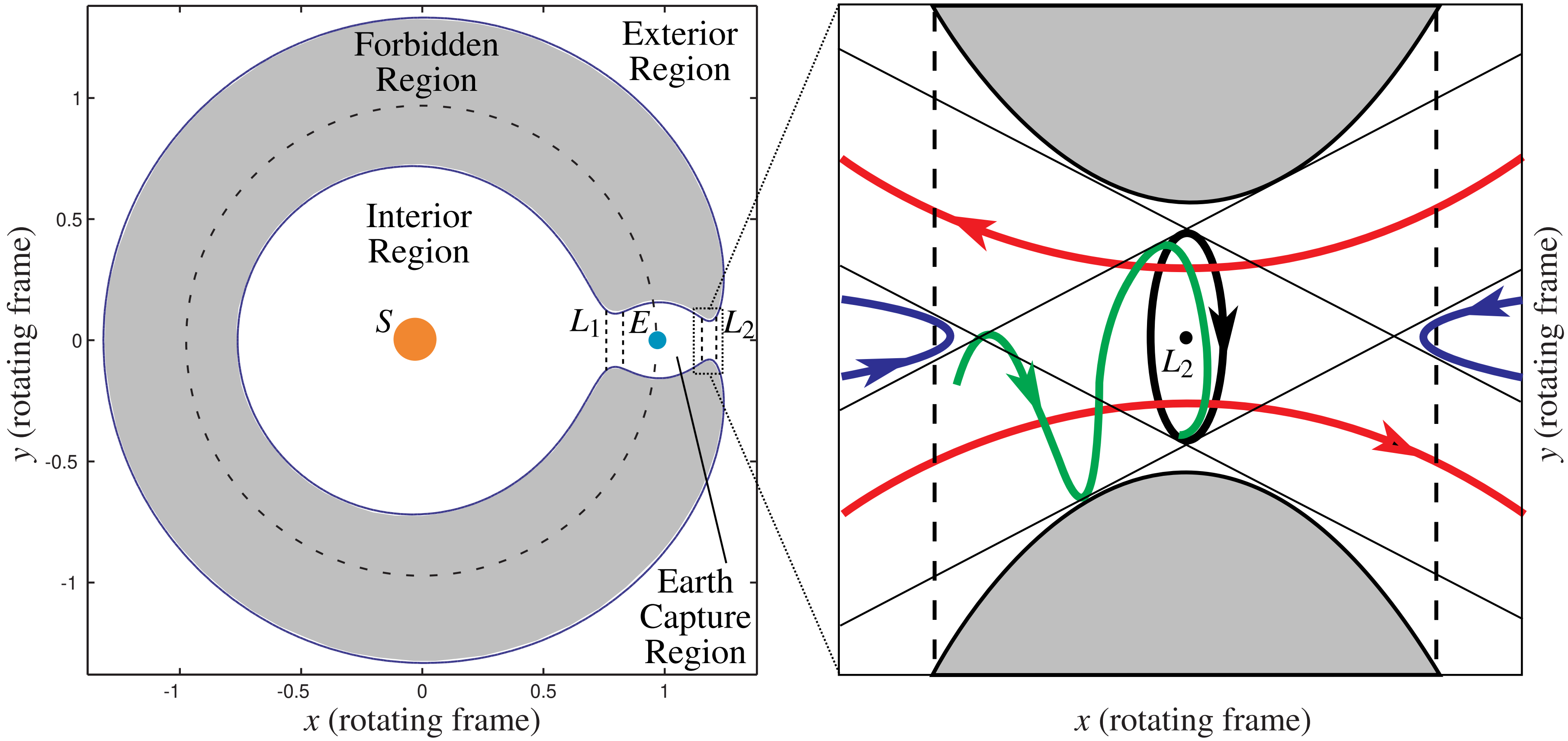
Control often makes unstable objects, not attractors, of interest![1] Under proper ***control management*** they are incredibly ***energy efficient***.

[1]This is related to ***control of chaos***; see Bloch, A.M. and J.E. Marsden [1989] Controlling homoclinic orbits, *Theor. & Comp. Fluid Mech.* **1**, 179–190.

Hill's Regions

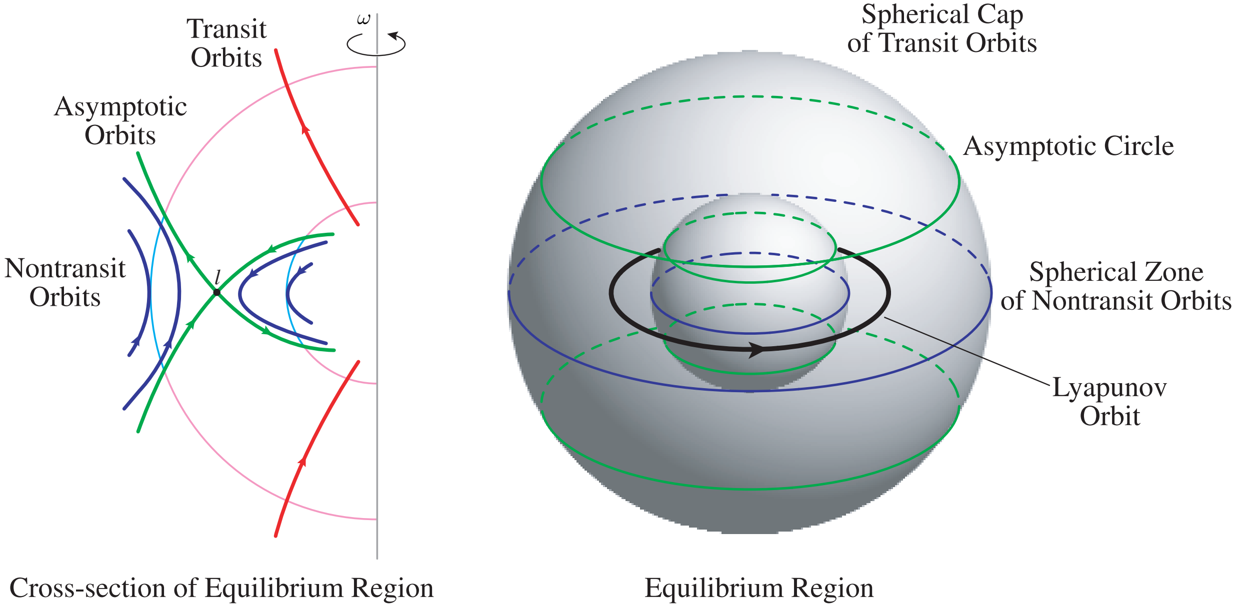
- Our main concern is the behavior of orbits whose energy is ***just above*** that of L_2. Roughly, we refer to this small energy range as the ***temporary capture energy*** range.
  - Fortunately, the temporary capture energy surfaces for the Sun-Earth and Earth-Moon systems intersect in phase space, making a fuel efficient transfer possible.
- The ***Hill's region*** is the projection of this energy region onto position space.
- The region not accessible for these energies is called the ***forbidden region***.
- For this case, the Hill's region contains a "neck" about L_1 and L_2. This equilibrium neck region and its relation to the global orbit structure is critical: it was studied in detail by ***Conley, McGehee and the Barcelona group***.

- Orbits with energy just above that of L_2 can be **transit orbits**, passing through the neck region between the ***exterior region*** (outside Earth's orbit) and the ***Earth temporary capture region*** (bubble surrounding Earth). They can also be **nontransit orbits** or **asymptotic orbits**.

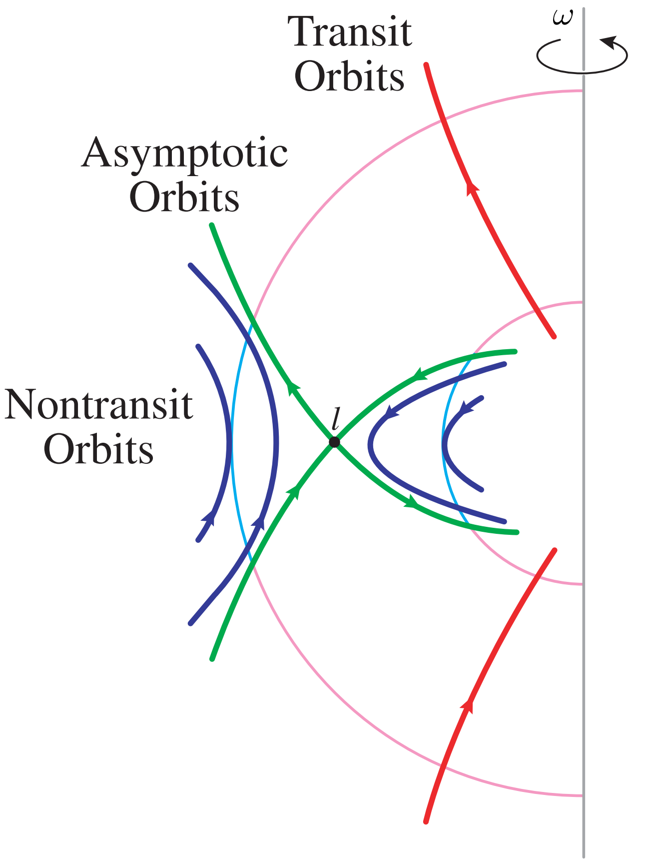


Equilibrium Region near a Lagrange Point

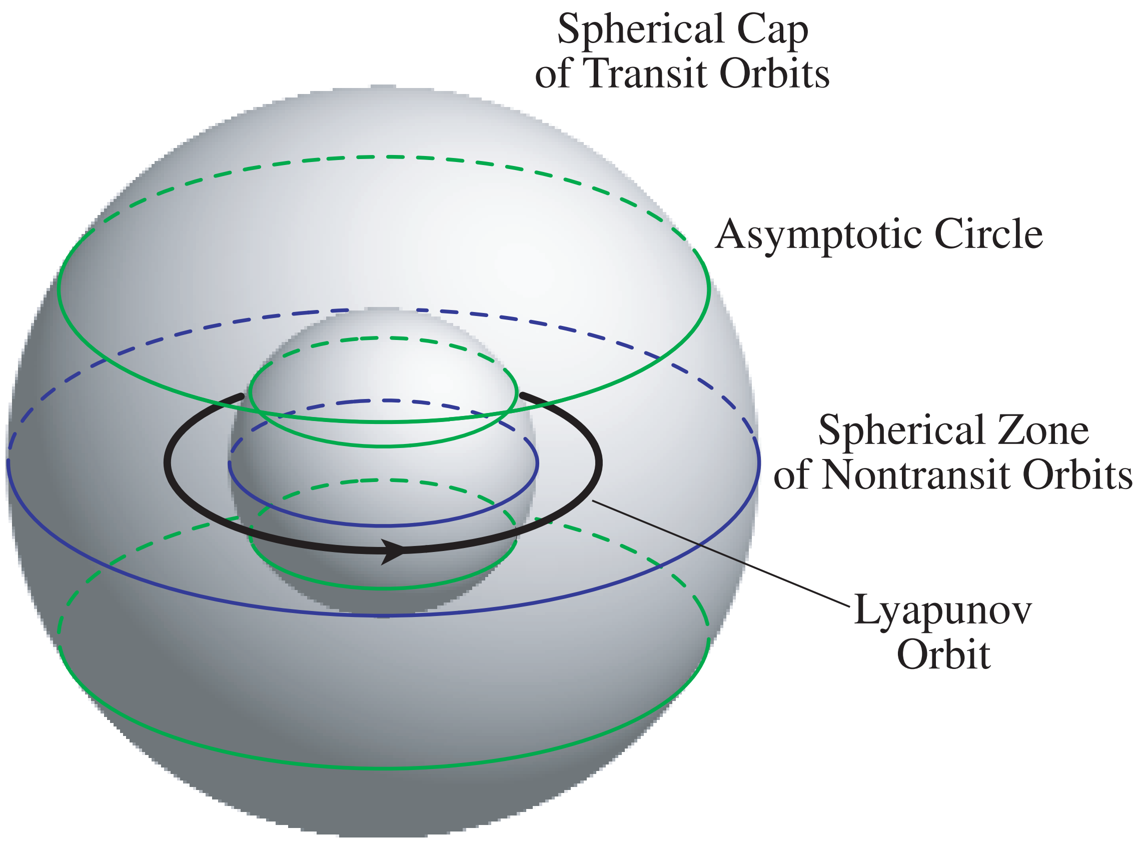
- **4 types** of orbits in the ***equilibrium region***.
  - **Black** circle is the unstable **periodic** Lyapunov orbit.
  - 4 cylinders of **asymptotic** orbits form pieces of stable and unstable manifolds. They intersect the bounding spheres at asymptotic circles, separating ***spherical caps***, which contain **transit** orbits, from ***spherical zones***, which contain **nontransit** orbits.



- Roughly speaking, the equilibrium region has the dynamics of a ***saddle* $\times$ *harmonic oscillator***.



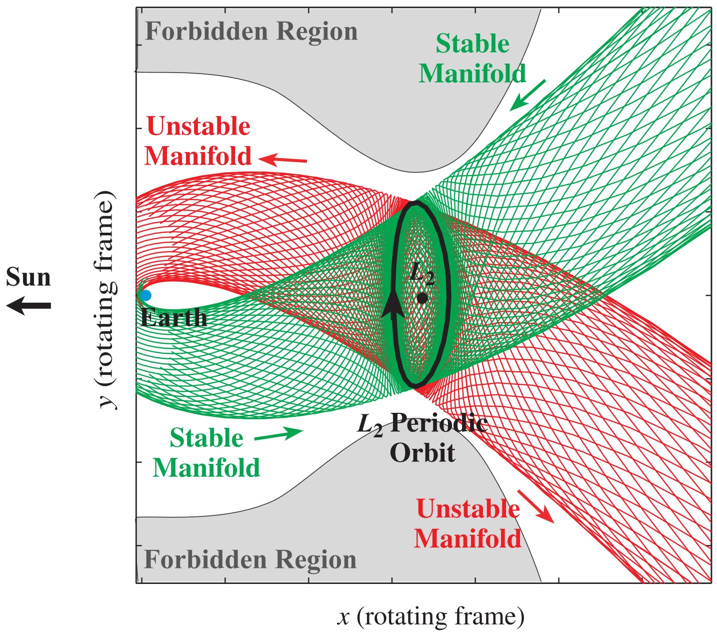
Cross-section of Equilibrium Region



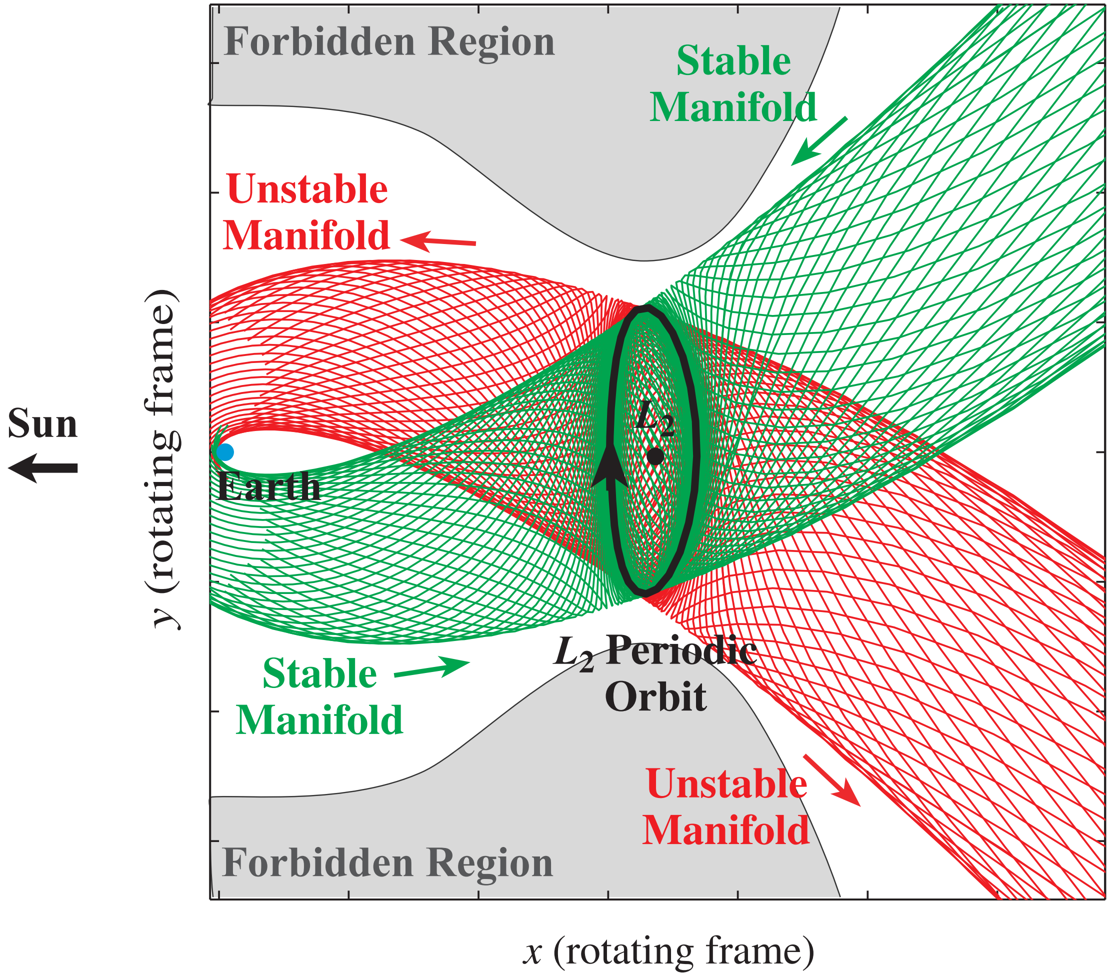
Equilibrium Region

Tubes Partition the Energy Surface

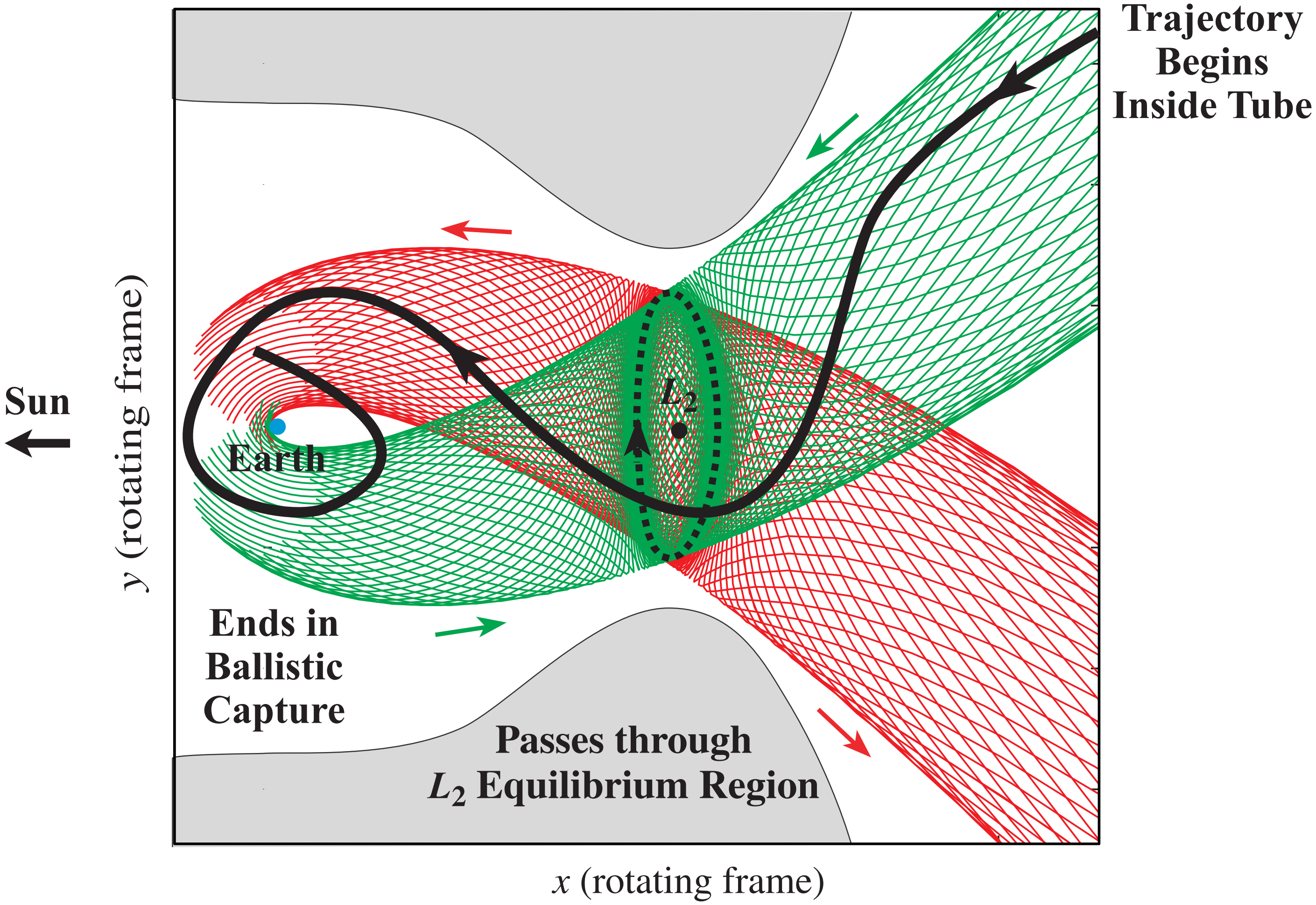
- Stable and **unstable** ***manifold tubes*** act as ***separatrices*** for the flow in the equilibrium region.
  - Those inside the tubes are ***transit*** orbits.
  - Those outside the tubes are ***nontransit*** orbits.
  - e.g., transit from outside Earth's orbit to Earth capture region possible ***only*** through L_2 periodic orbit stable tube.



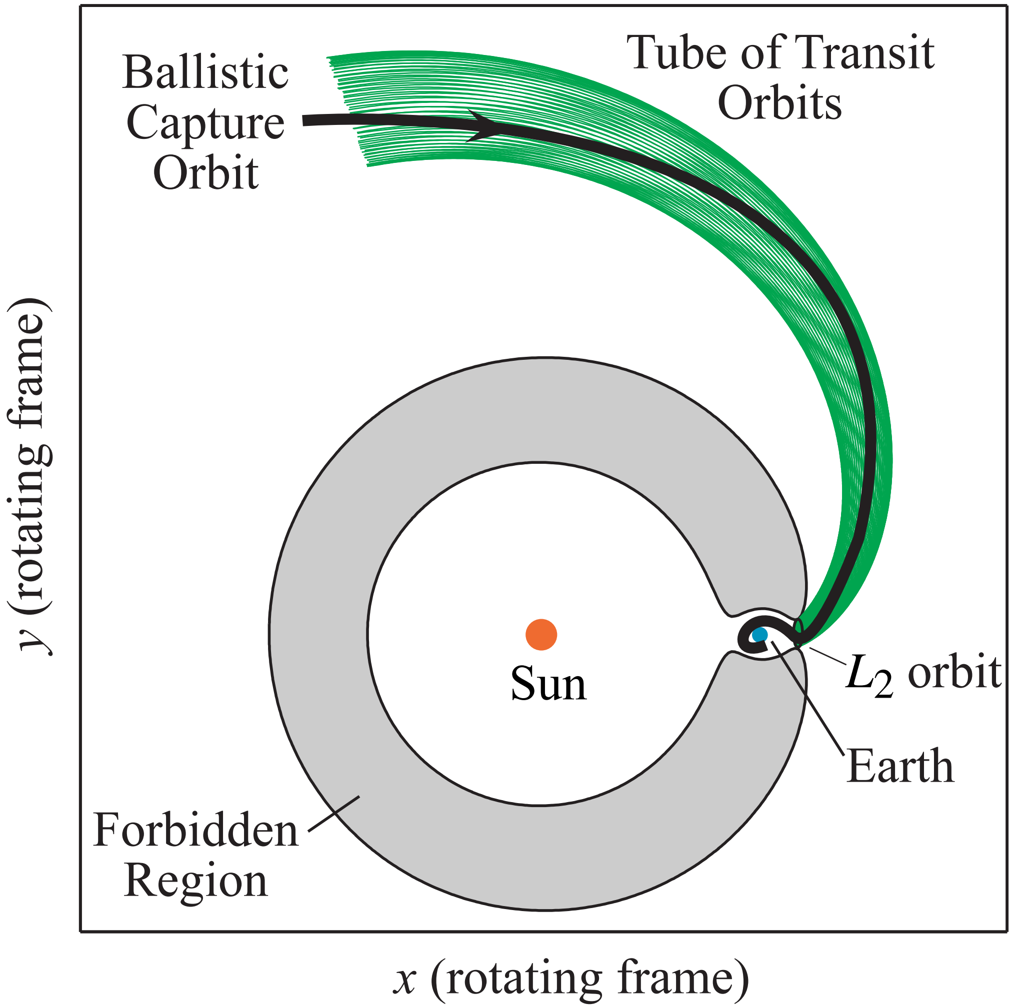
- Stable and **unstable** manifold tubes effect the ***transport*** of material to and from the capture region.



- Tubes of transit orbits can be utilized for ***ballistic capture***.



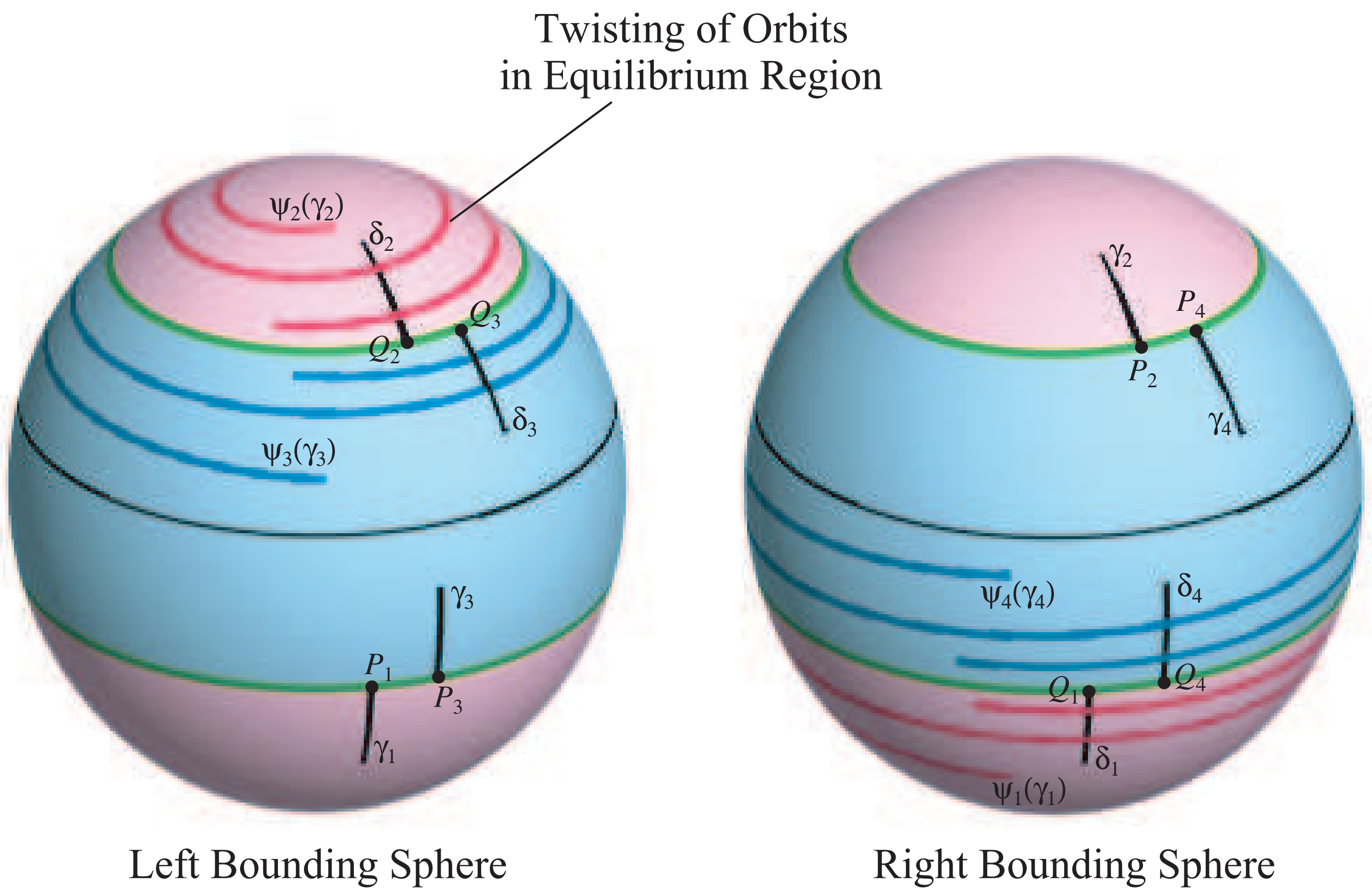
- Invariant manifold tubes are ***global objects*** — extend far beyond vicinity of Lagrange points.



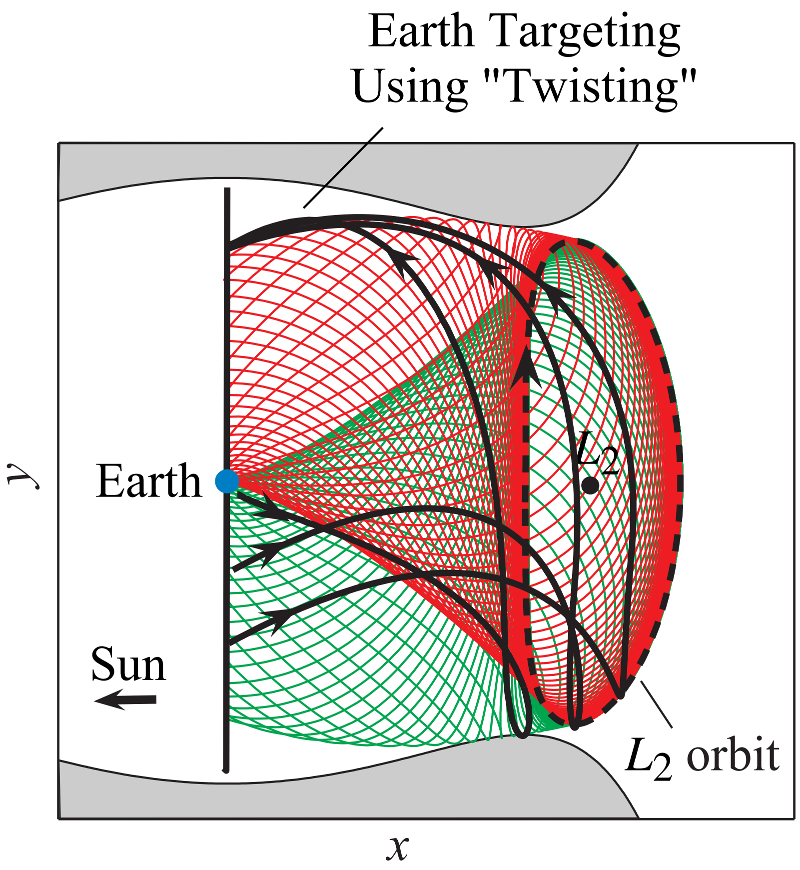
Twisting of Orbits in the Equilibrium Region

- Recall that the equilibrium region roughly has the dynamics of a saddle $\times$ harmonic oscillator.
  - Orbits ***twist*** in the equilibrium region, roughly following the Lyapunov orbit.
  - The closer the approach to the Lyapunov orbit, the more the orbit twists.
  - Thus, the closer an orbit begins to the tube on its approach to the equilibrium region, the more it will be twisted when it exits the equilibrium region.
- This twisting near the tubes will be used in the Earth backward targeting portion of the Earth-to-Moon trajectory.

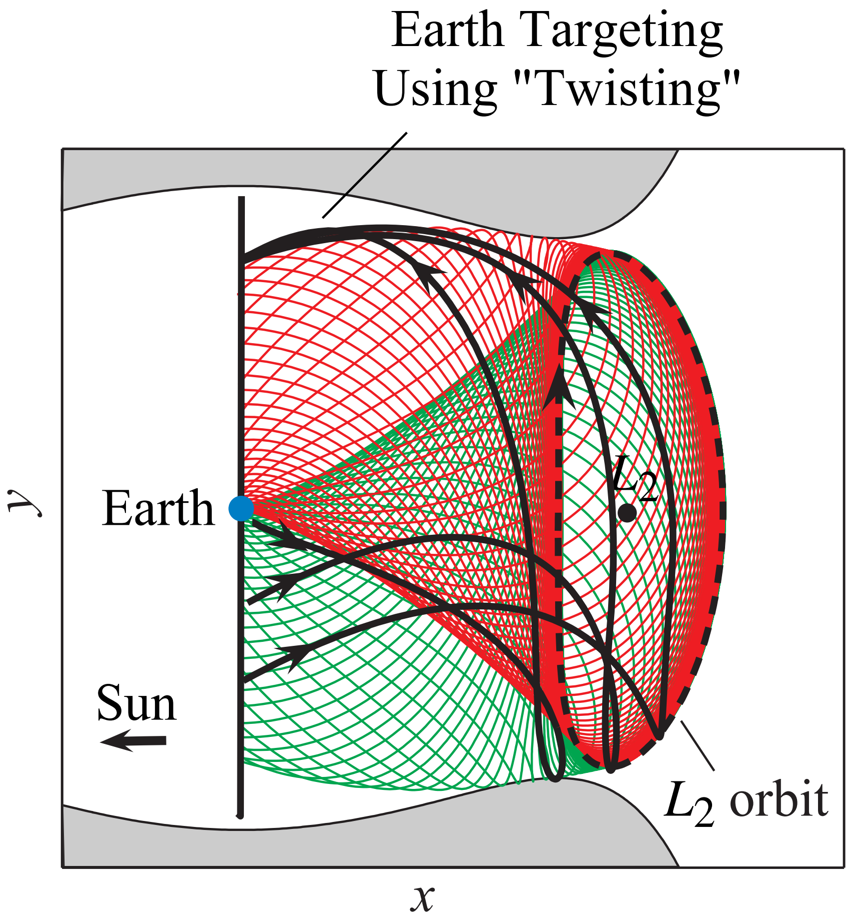
- Twisting in the equilibrium region can be understood in terms of mappings between the bounding spheres.



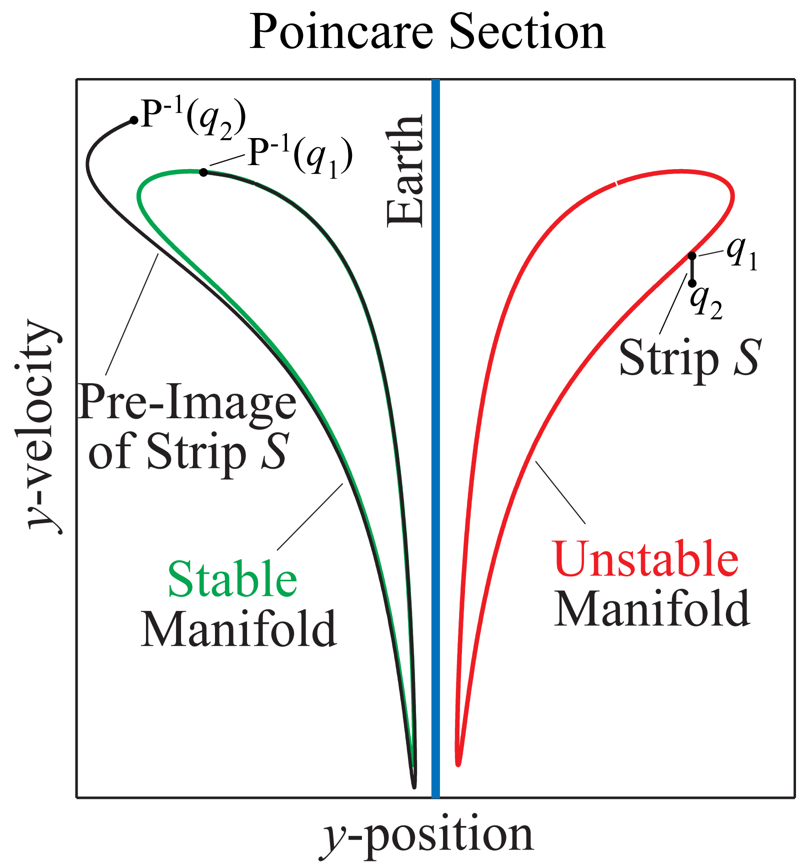
- We select a line of constant x-position passing through the Earth in the Earth capture region.
- Pick a few points near the **unstable tube**, with the same position but slightly different velocities.

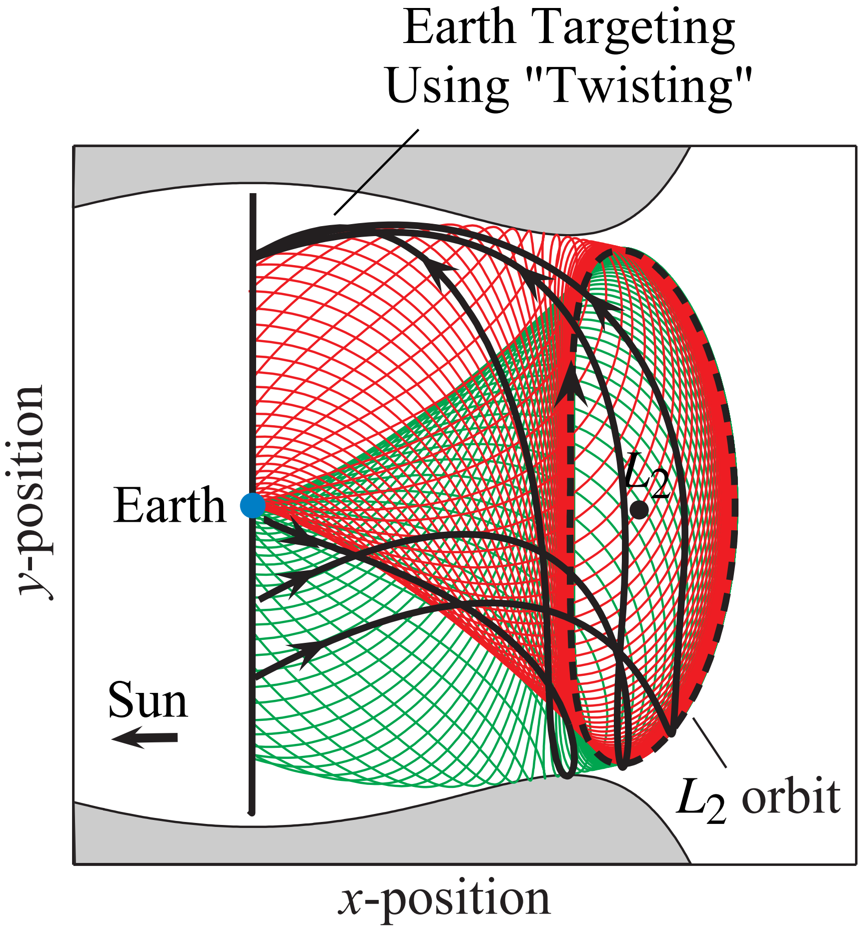


- With a slight change in the velocity, we can target any position on the lower line, where the mirror image **stable tube** intersects.



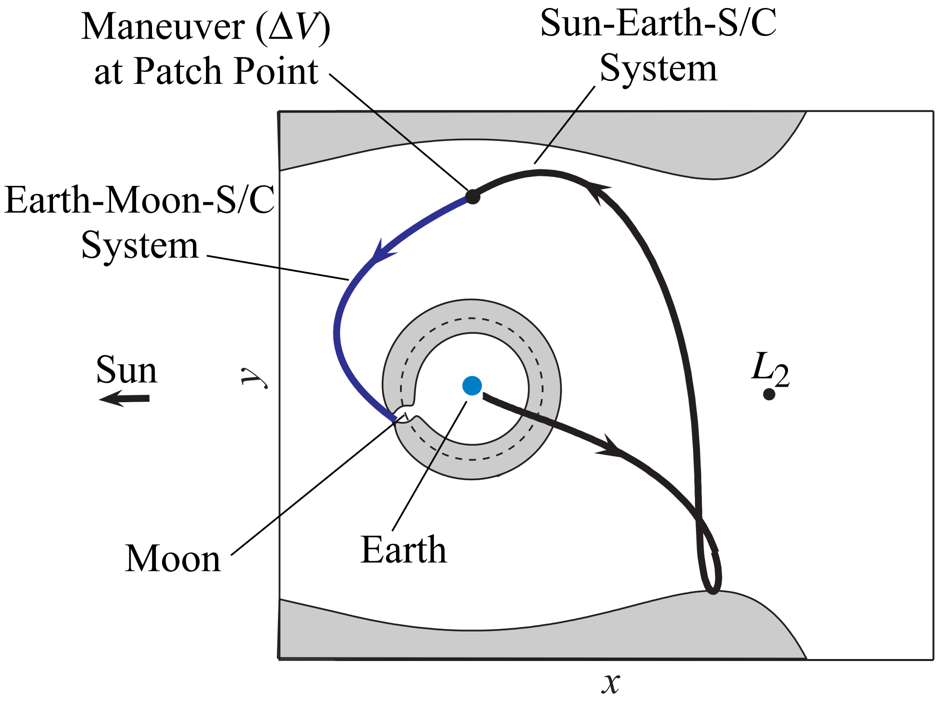
- Look at Poincaré section along this line of constant x-position.
- With infinitesimal changes in velocity, any point near lower tube cross-section can be targeted.





Two Coupled 3-Body Systems

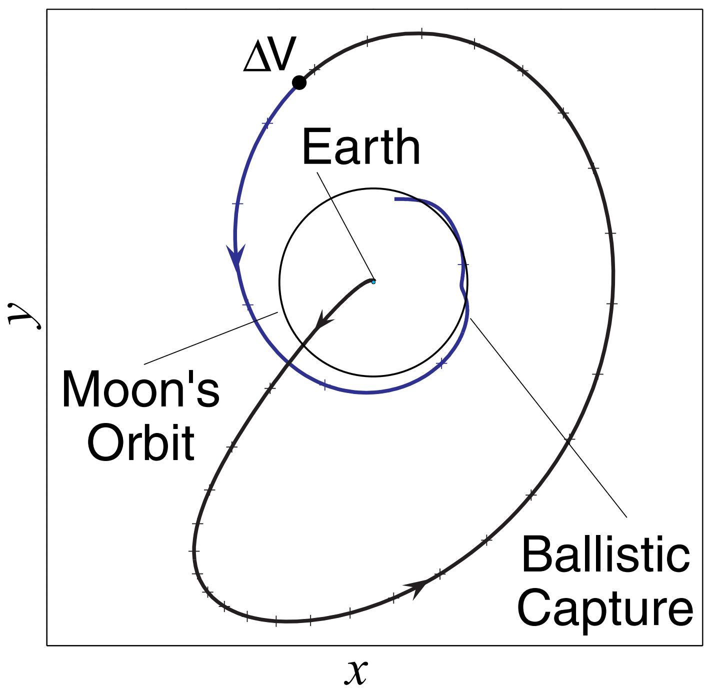
- We obtain the fuel efficient tranfer by taking full advantage of the dynamics of the 4-body system (Earth, Moon, Sun, and spacecraft) by initially modeling it as ***two coupled planar circular restricted 3-body systems***.
- In this approach, we utilize the Lagrange point dynamics of both the Earth-Moon-spacecraft and Sun-Earth-spacecraft systems.



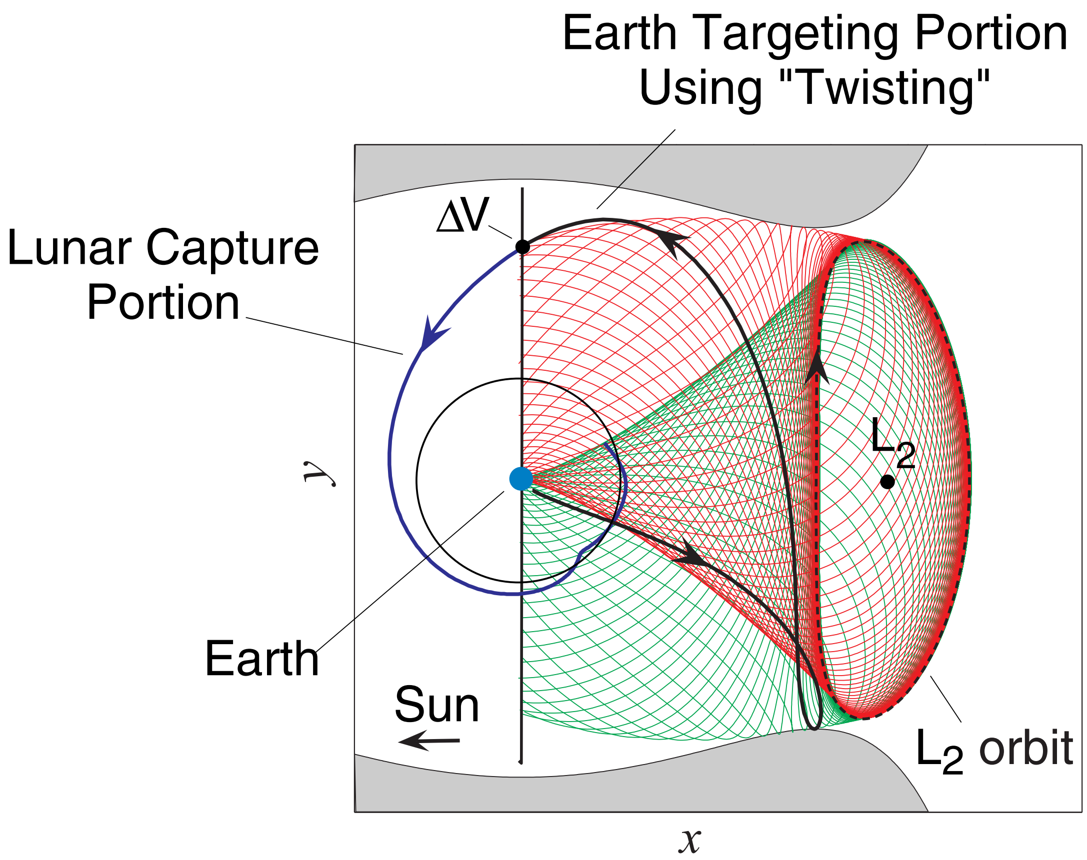
- In this simplified model, the Moon is on a circular orbit about the Earth, the Earth (or rather the Earth-Moon center of mass) is on a circular orbit about the Sun, and the systems are coplanar.
- In the actual solar system:
  - Moon's eccentricity is 0.055
  - Earth's eccentricity is 0.017
  - Moon's orbit is inclined to Earth's orbit by 5°
- These values are low, so the coupled planar circular 3-body problem is considered a ***good starting model***.
- An orbit which becomes a real mission is typically obtained first in such an approximate system and then later refined through more precise models which inlude effects such as out-of-plane motion, eccentricity, the other planets, solar wind, etc.
- However, tremendous insight is gained by considering a simpler model which reveals the ***essence of the transfer dynamics***.

- Notice that in the Sun-Earth rotating frame, patterns such as the Sun-Earth L_2 portion of the trajectory are made plain which are not discernable in the inertial frame.

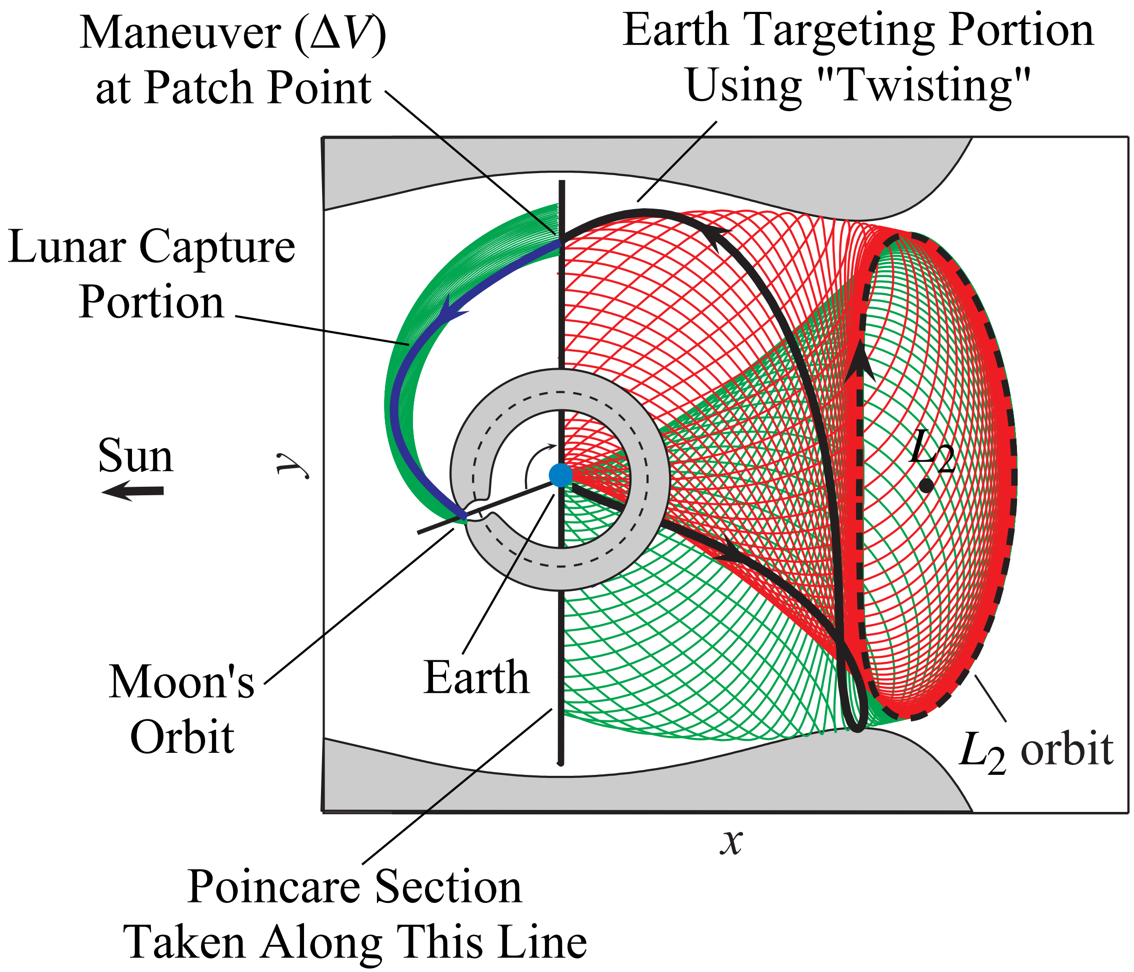
Inertial Frame



Sun-Earth Rotating Frame



- The construction is done mainly in the Sun-Earth rotating frame using the Poincaré section (that passes through the x-position of the Earth) which help to glue the Sun-Earth Lagrange point orbit portion of the trajectory with the lunar ballistic capture portion.

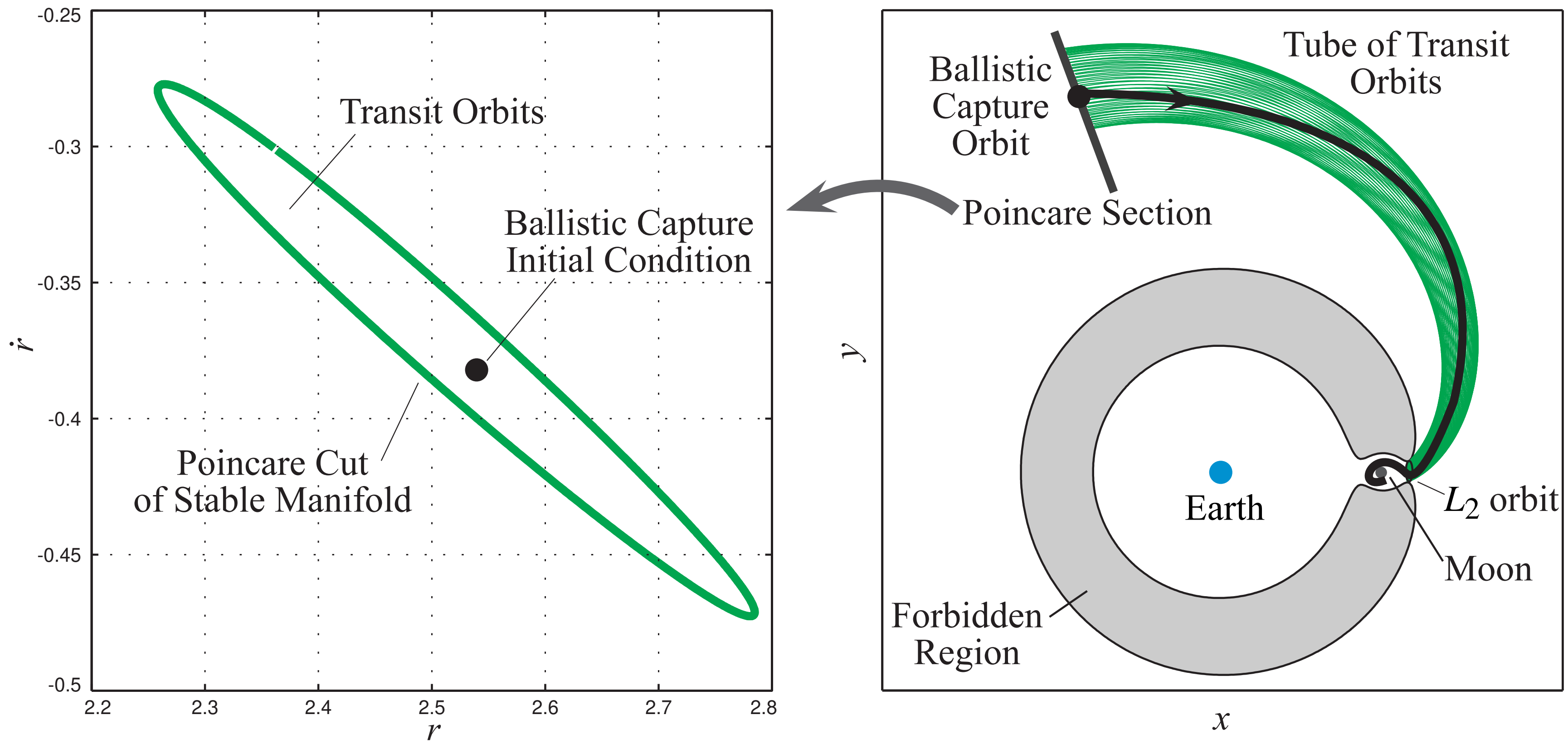


Construction of Earth-to-Moon Transfer

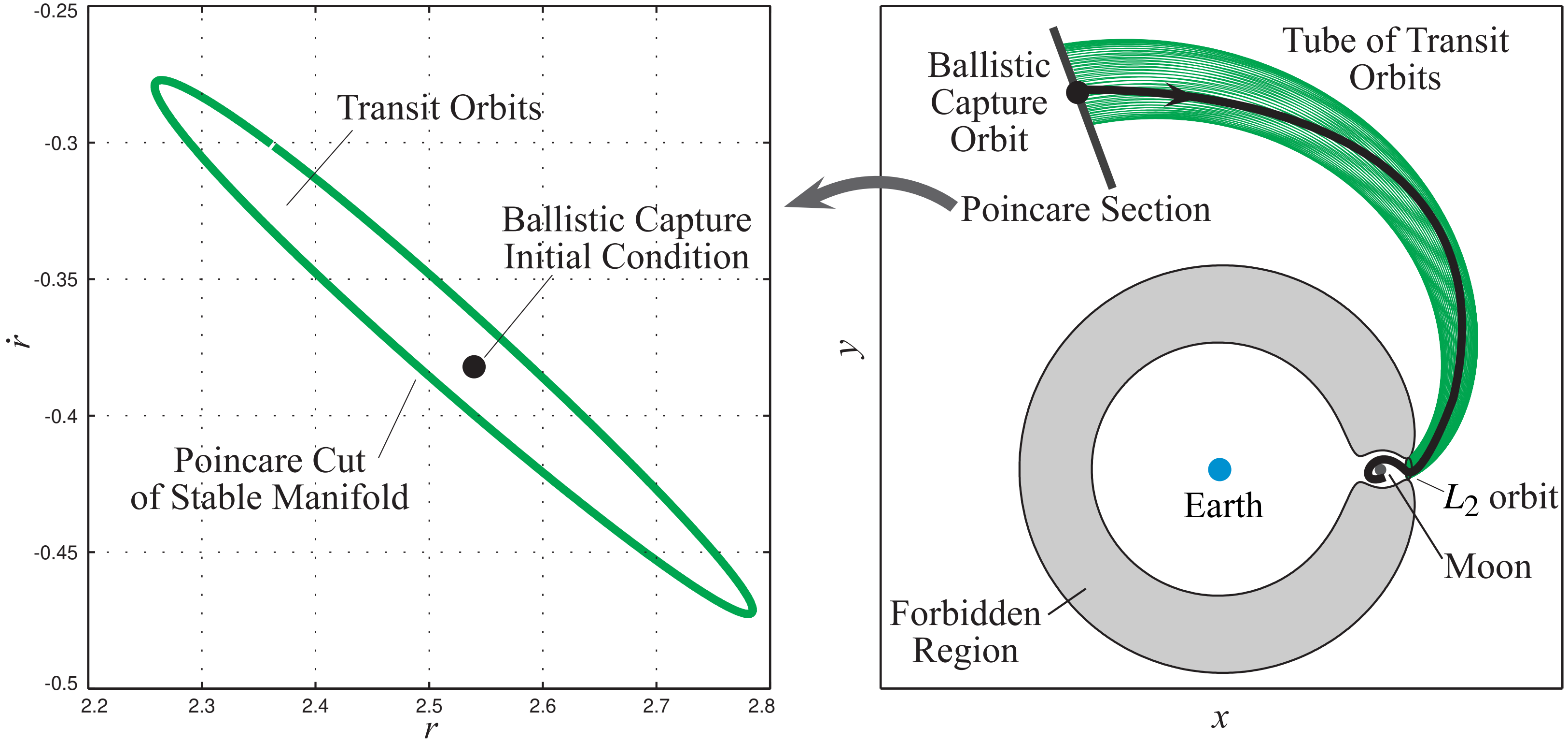
- ***Strategy***: Find an initial condition (position and velocity) for a spacecraft on the Poincaré section such that:
  - when integrating ***forward***, spacecraft will be guided by ***Earth-Moon manifold*** and get ballistically captured by the Moon;
  - when integrating ***backward***, spacecraft will hug ***Sun-Earth manifolds*** and return to Earth.
- We utilize two important properties of Lagrange point dynamics:
  - Tube of ***transit orbits*** is key in finding capture orbit for Earth-Moon portion of the design;
  - ***Twisting*** of orbits in equilibrium region is key in finding a fuel efficient transfer for Sun-Earth portion of the design.

Lunar Ballistic Capture Portion

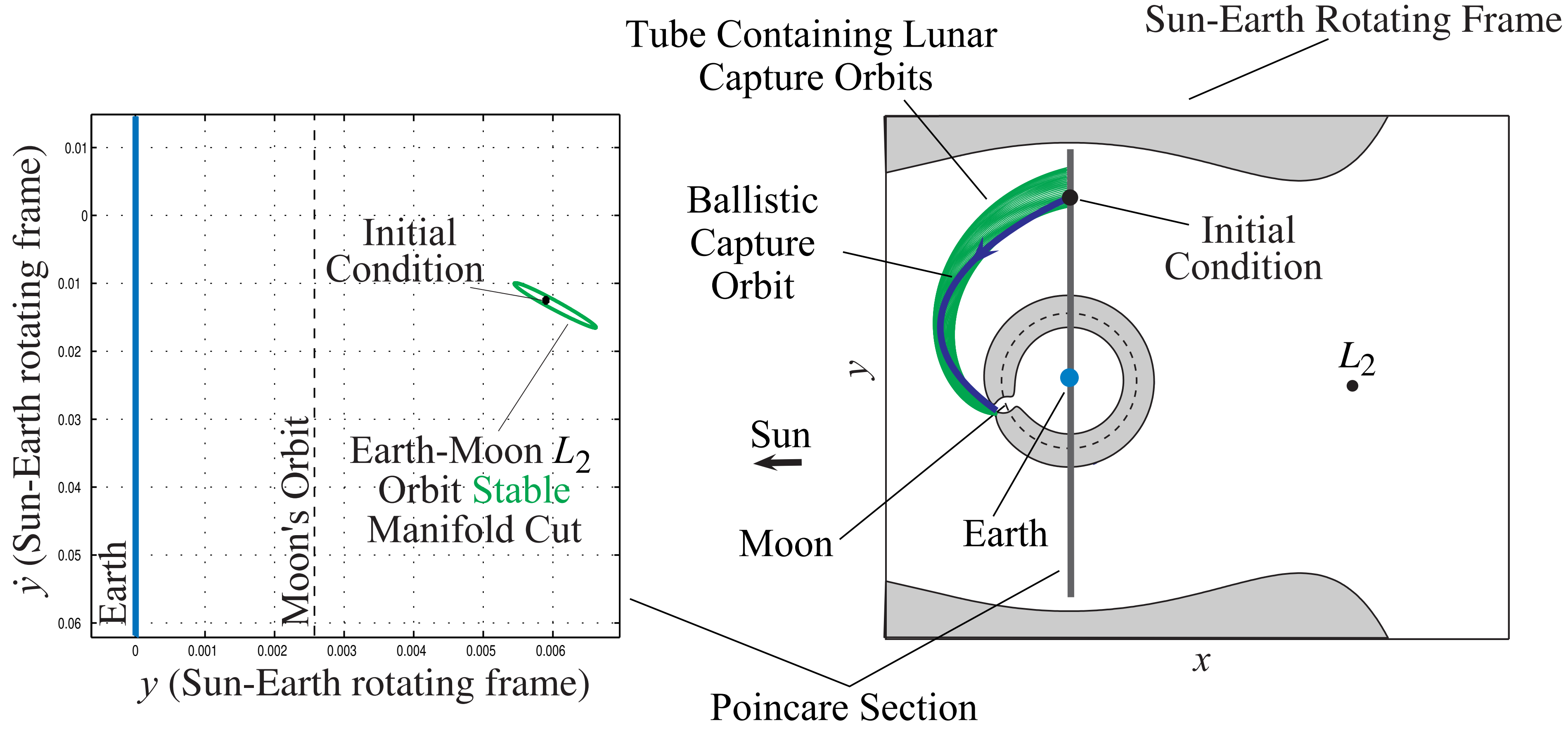
- Recall that by targeting the region enclosed by the **stable manifold tube** of the Earth-Moon-S/C system's L_2 Lyapunov orbit, we can construct an orbit which will get ballistically captured by the Moon.



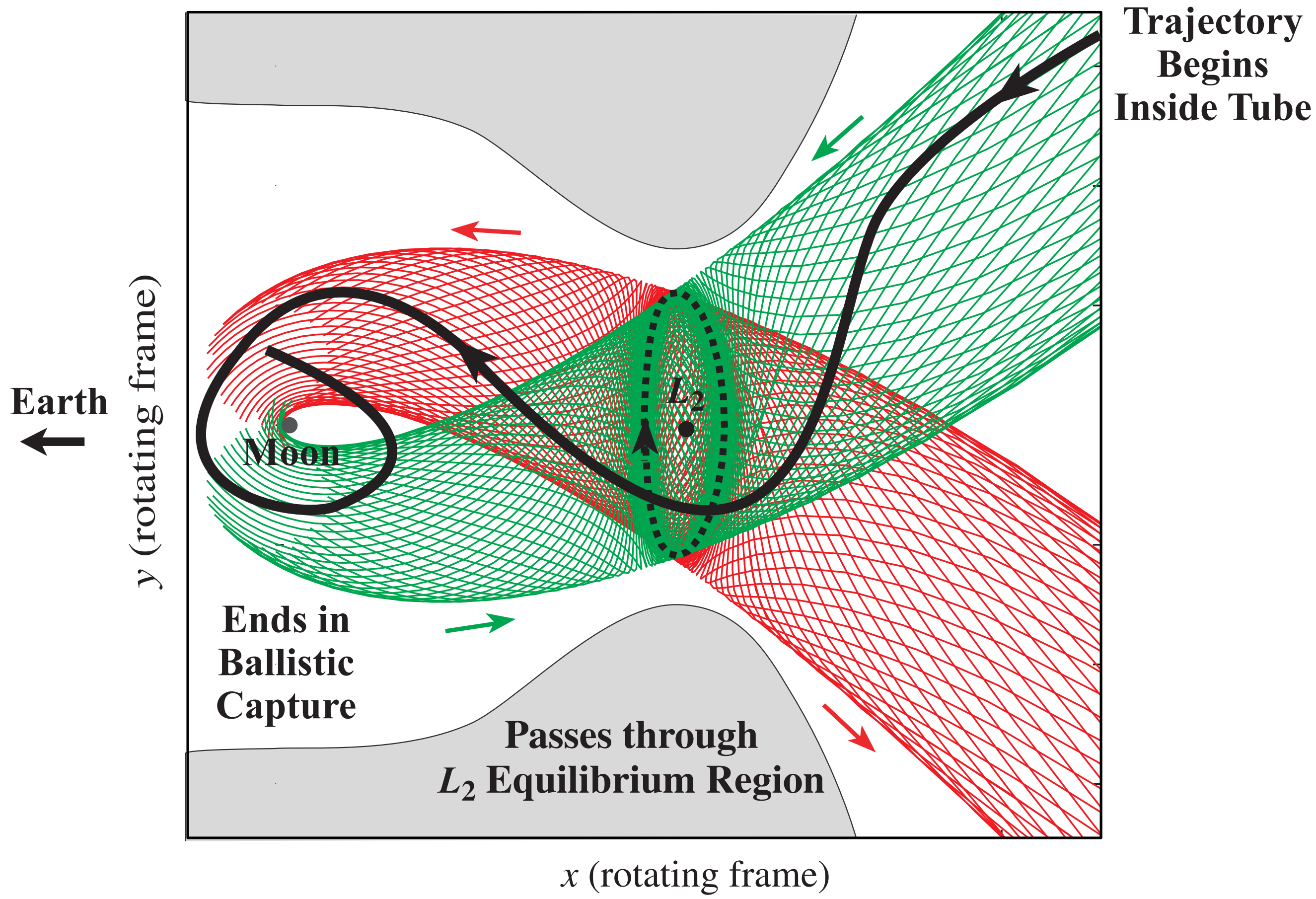
- When we transform the Poincaré section of the **stable manifold** of the Lyapunov orbit about the Earth-Moon L_2 point into the Sun-Earth rotating frame, we obtain a curve. A point interior to this curve, with the correct phasing of the Moon, will approach the Moon when integrated forward. The ***phasing*** of the Moon is determined by the orientation of the Poincaré section in the Earth-Moon system.



- Assuming the Sun is a negligible perturbation to the Earth-Moon-S/C 3-body dynamics, any spacecraft with initial conditions within this **closed loop** will be ballistically captured by the Moon.

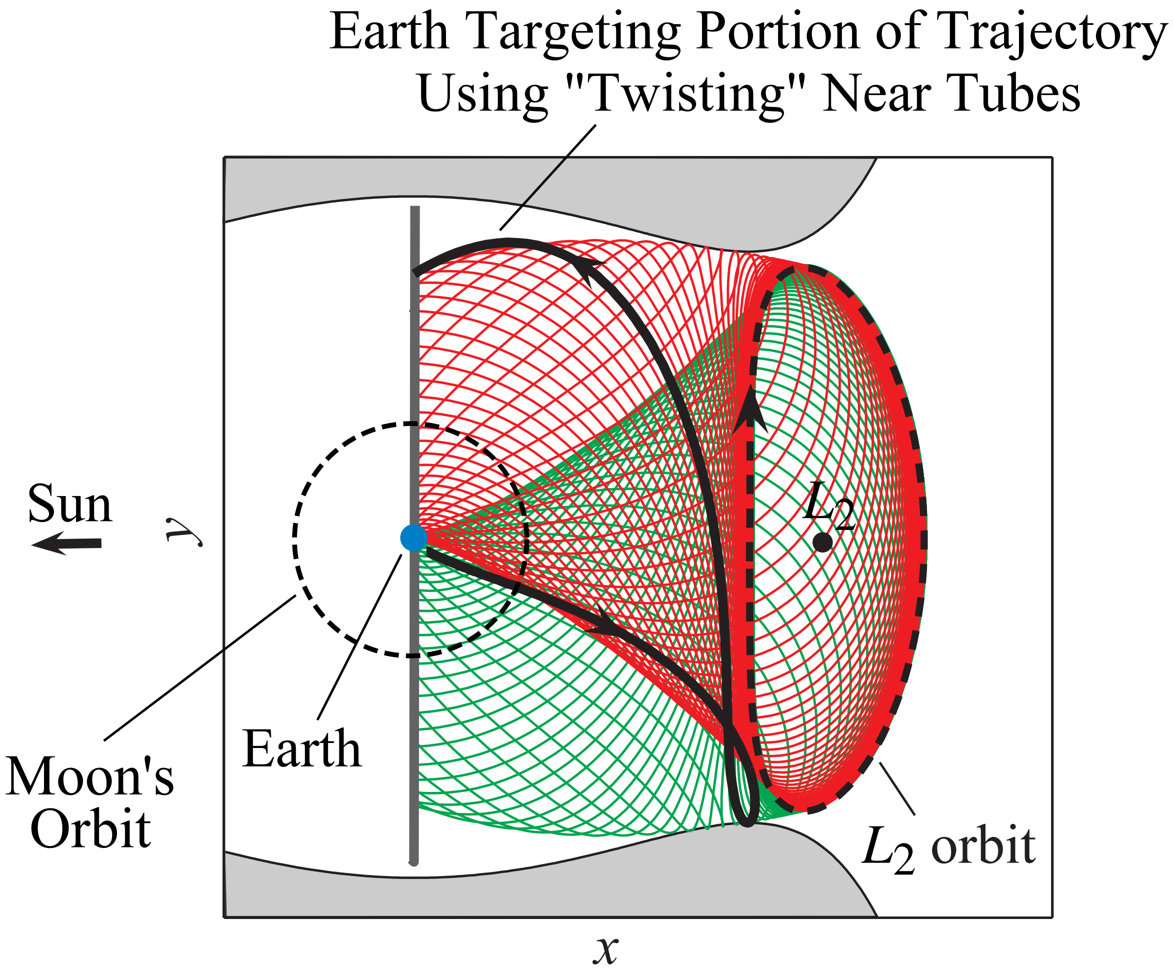


- "Ballistic capture by the Moon" means an orbit which under natural dynamics gets within the sphere of influence of the Moon (20,000 km) and performs at least one revolution around the Moon. In such a state, a slight ΔV will result in a stable capture (closing off the necks at L_1 and L_2).

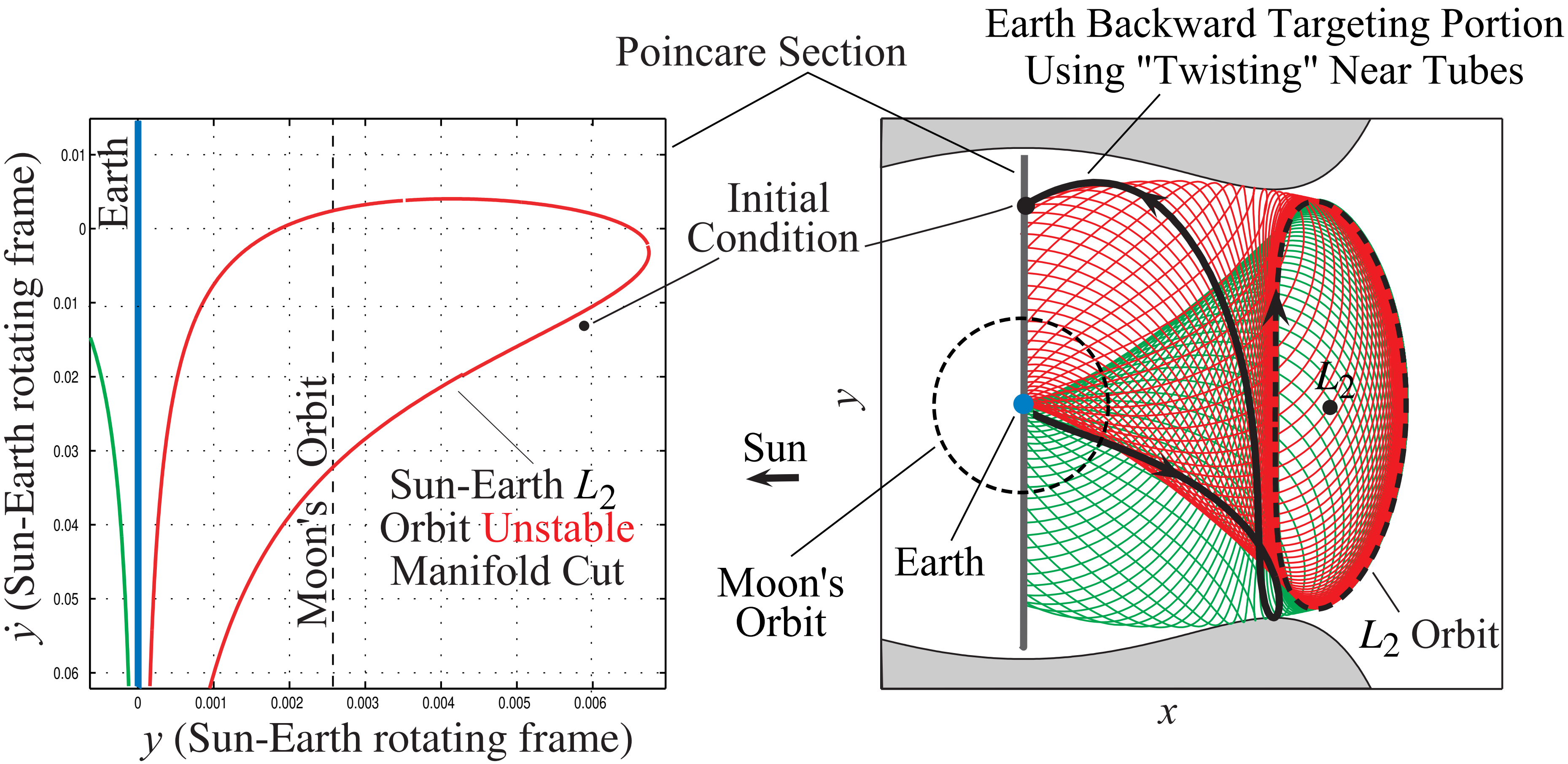


Earth Backward Targeting Portion

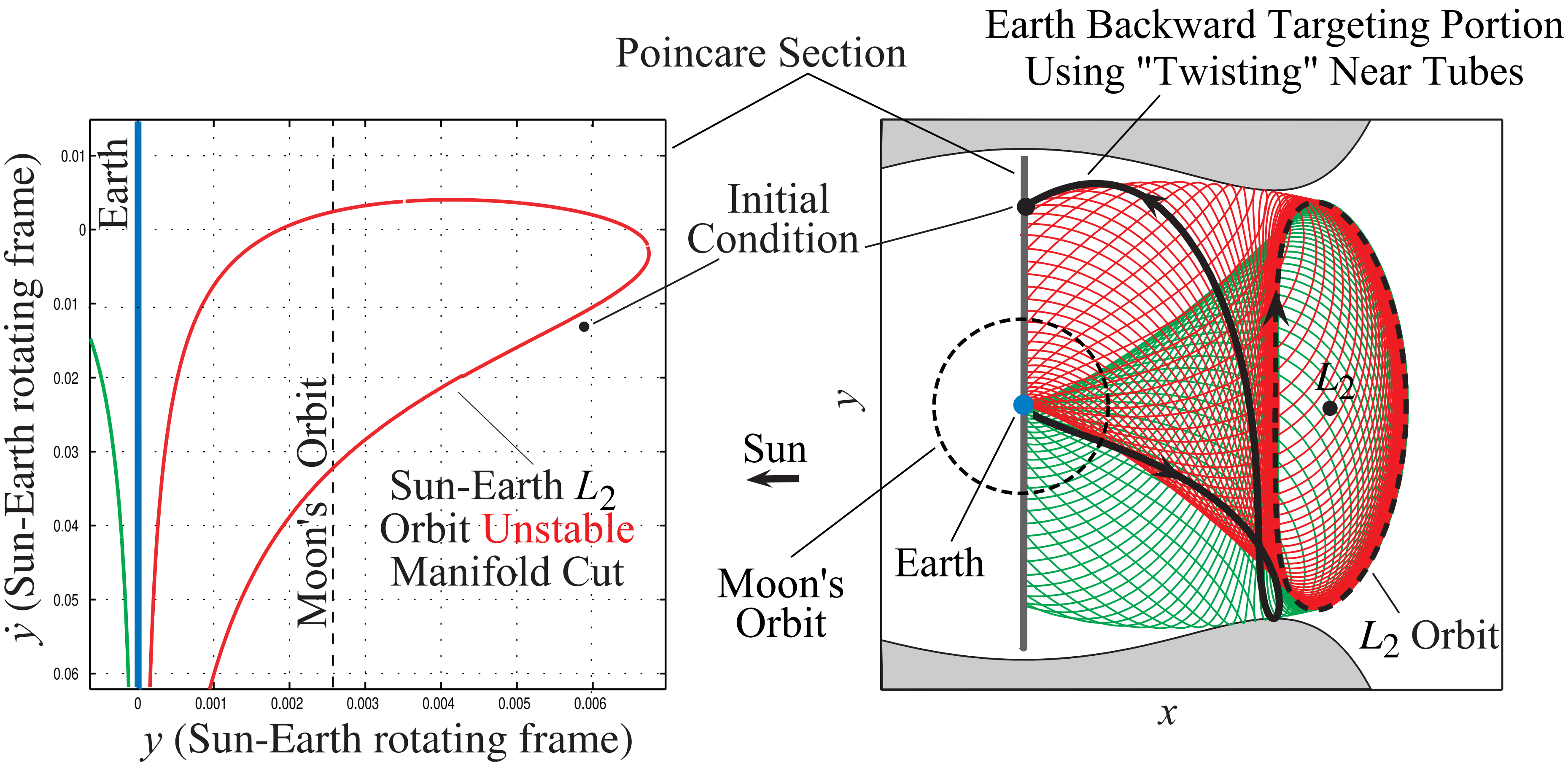
- Pick an energy in the temporary capture range of the Sun-Earth system which has L_2 orbit manifolds that come near a 200 km altitude Earth parking orbit.



- Compute Poincaré section along line of constant x-position passing through Earth:
  - The **red curve** is the Poincaré cut of the **unstable manifold** of the Lyapunov orbit around the Sun-Earth L_2.

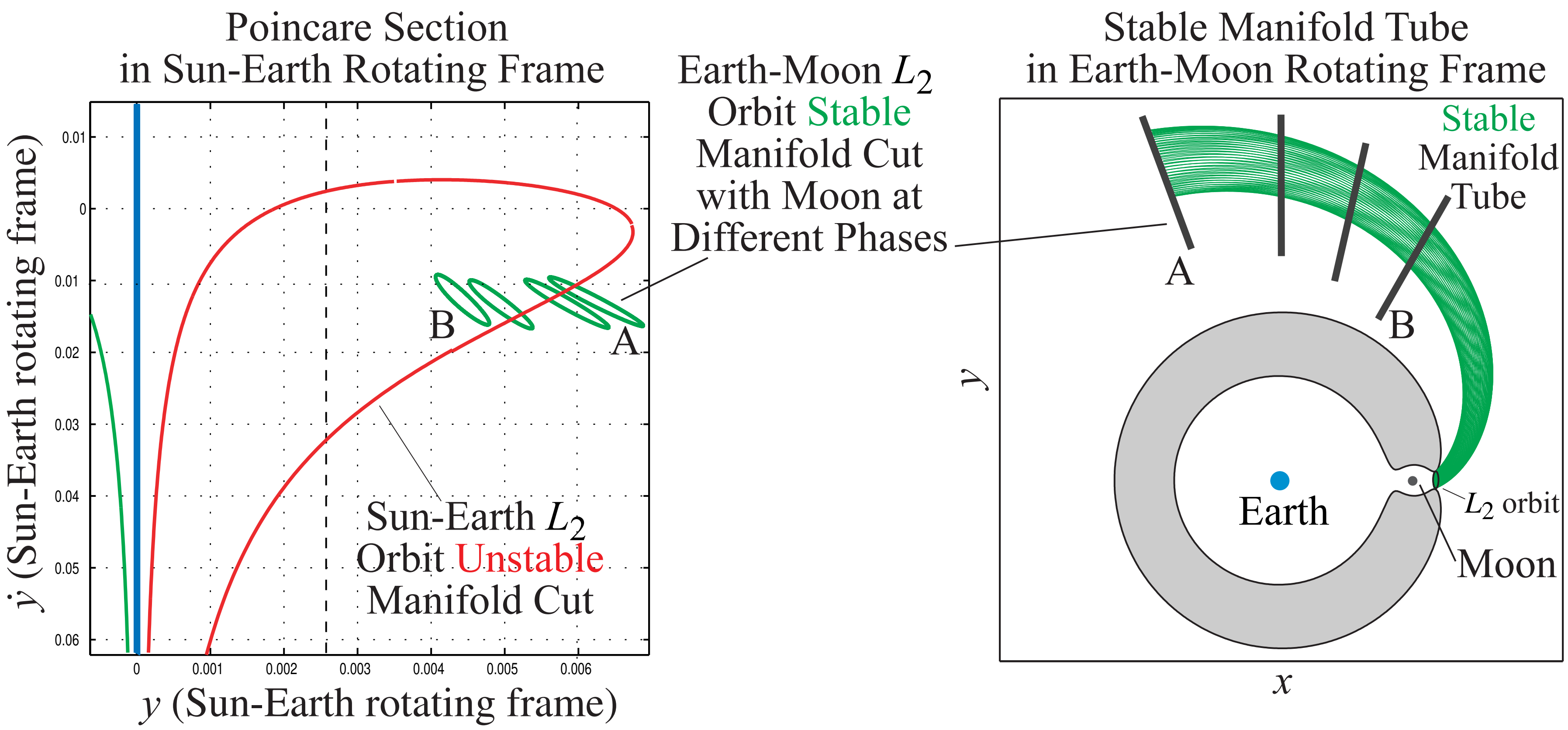


- Picking an initial condition just outside this curve, we can backward integrate to produce a trajectory coming back to the Earth parking orbit.

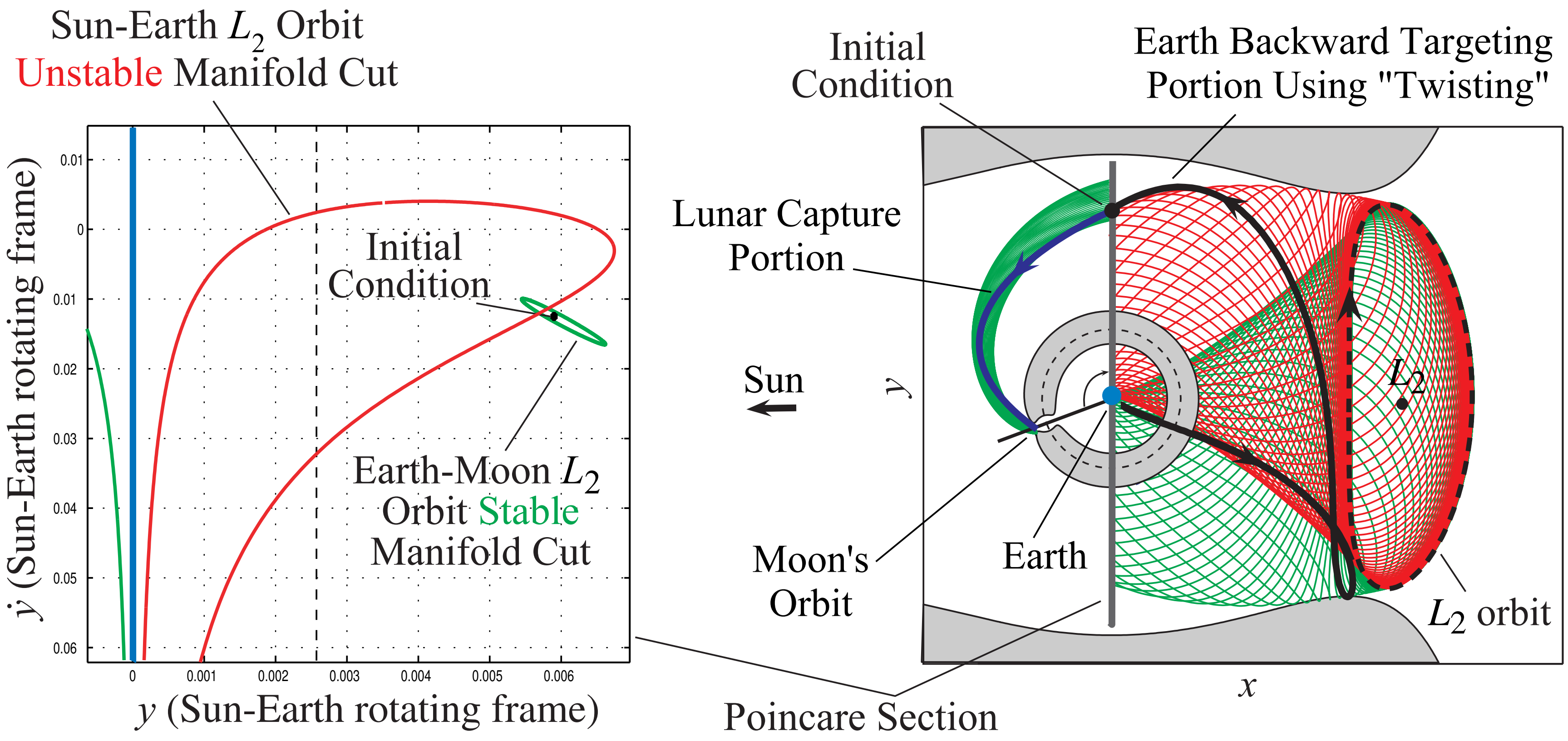


Connecting the Two Portions

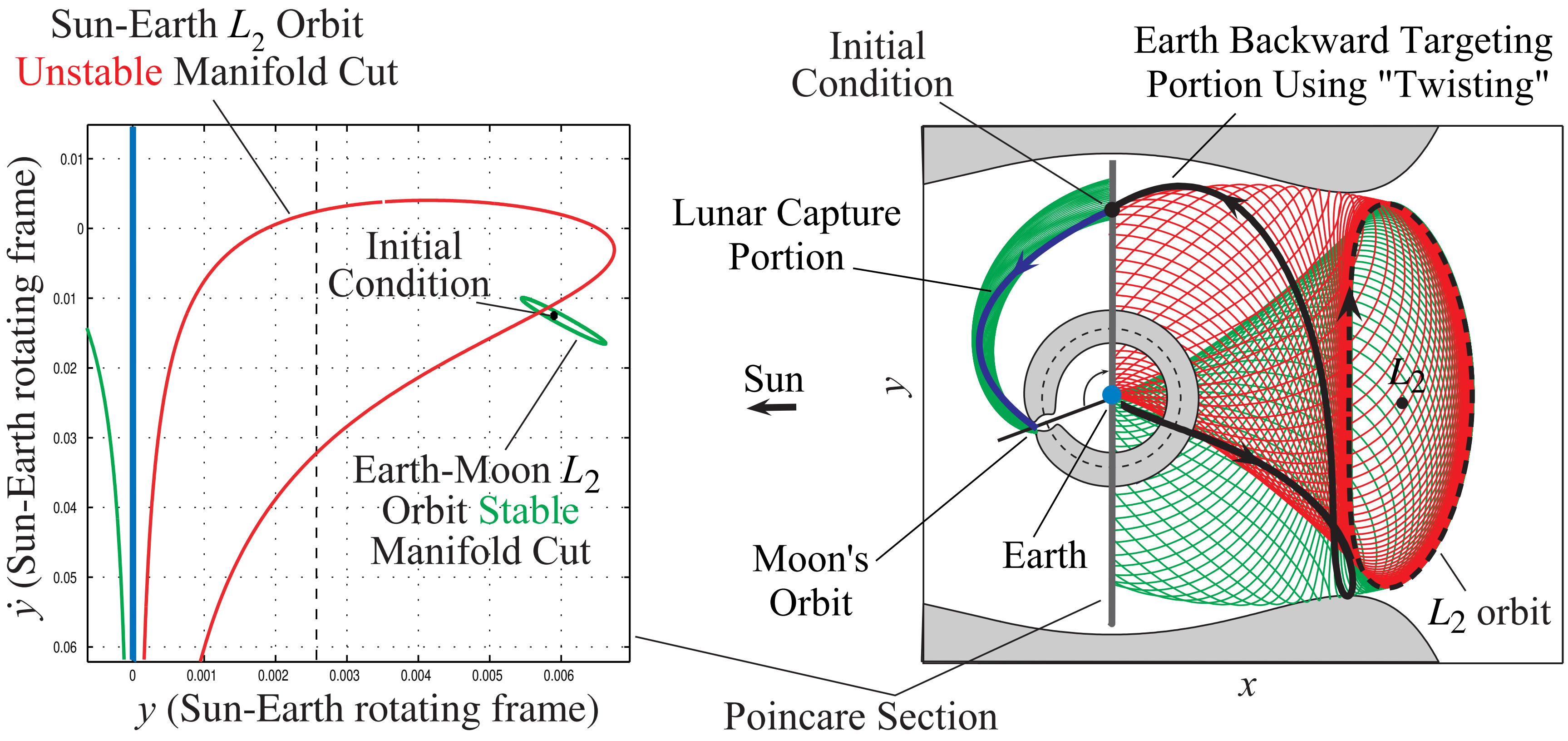
- Vary the phase of the Moon until **Earth-Moon L_2 manifold curve** intersects **Sun-Earth L_2 manifold curve**.



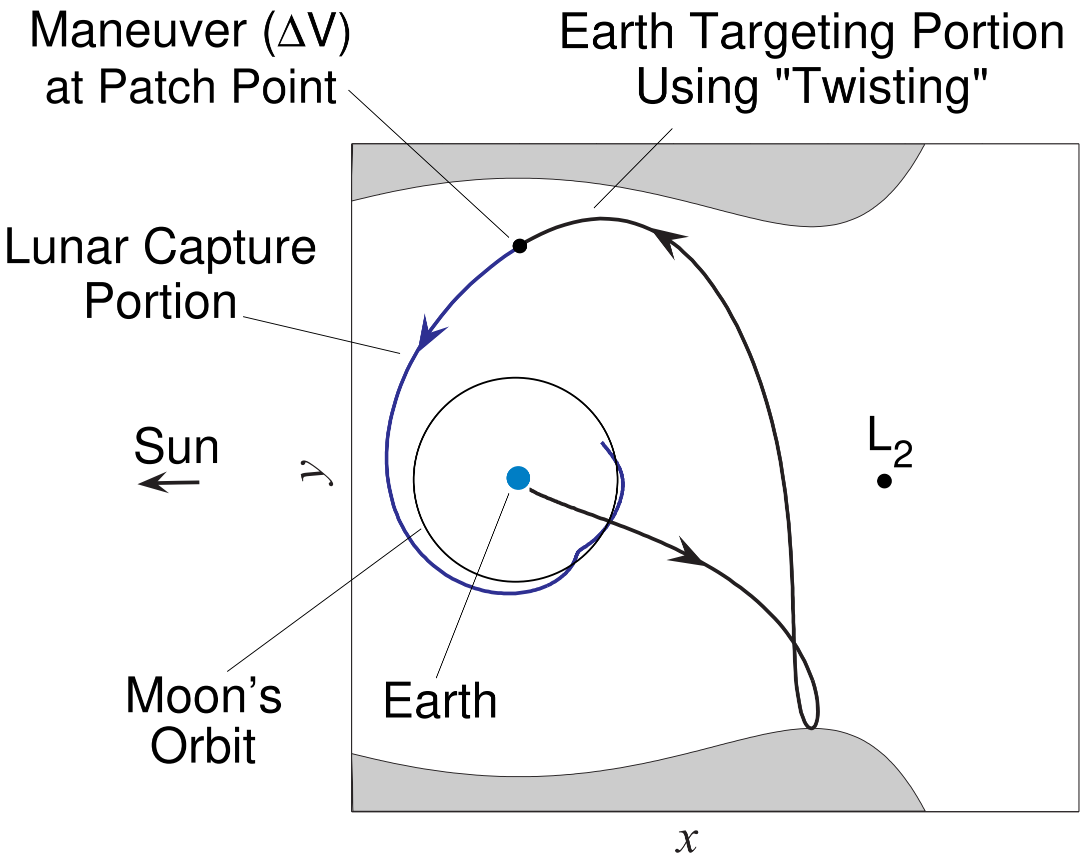
- Intersection is found!
- In the region which is in the interior of the **green curve** but in the exterior of the **red curves**,



- an orbit will get ballistically captured by the Moon when integrated forward;
- when integrated backward, orbit will hug the **unstable** manifold back to the Sun-Earth L_2 equilibrium region with a twist, and then hug the **stable** manifold back towards Earth.



- With only a slight modification, a midcourse ΔV at the patch point (34 m/s), this procedure produces a genuine solution integrated in the bicircular 4-body problem.
- Since capture at Moon is natural (zero ΔV), the amount of on-board fuel necessary is lowered (by about 20%).



Why Does It Work?

- Heuristic arguments for using the coupled 3-body model:
  - When outside the Moon's small sphere of influence (20,000 km), which is most of pre-capture flight, we can consider the Moon's perturbation on the Sun-Earth-S/C 3-body system to be negligible. Thus, can utilize Sun-Earth Lagrange point invariant manifold structures.
  - The midcourse ΔV is performed at a point where the spacecraft is entering the Earth's sphere of influence (900,000 km), where we can consider the Sun's perturbation on the Earth-Moon-S/C 3-body system to be negligible. Thus, Earth-Moon Lagrange point structueres can be utilized for capture sequence.

What Next? — Future Work

- Using differential correction and continuation, we expect this trajectory can be used as an initial guess to generalize to the three-dimensional 4-body problem as well as the full solar system model.
- Optimize trajectory (minimize total fuel consumption):
  - Incorporate initial lunar swingby
  - Apply optimal control (e.g., COOPT)
  - Use continuous thrust (low-thrust)
- Develop systematic procedure for coupling multiple 3-body systems. This will aid in the design of innovative space missions and help aid in understanding subtle non-Keplerian transport throughout the solar system.

More Information and References

- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] ***Heteroclinic connections between periodic orbits and resonance transitions in celestial mechanics***, *available from* **http://www.cds.caltech.edu/~marsden/**

- Koon, W.S., M.W. Lo, J.E. Marsden and S.D. Ross [2000] ***Shoot the Moon***, *available at request*

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